On the Normalization Condition for Cost of Living Comparisons under Time-Varying Preferences*

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Abstract

There is growing interest in measuring inflation in the presence of time-varying preferences. To make price comparisons under changing preferences, a number of studies are imposing normalization conditions on preference parameters, assuming cardinal utility functions. The resulting price indexes depend on the choice of normalization condition imposed, necessitating a careful specification of this condition. Carluccio et al. (2023) adopt a normalization where the arithmetic mean of the time-varying taste parameters remains constant, whereas Hottman et al. (2016) and Redding and Weinstein (2020) maintain a constant geometric mean. In this paper we invoke the commensurability axiom which requires the price index to be independent of units of measurement. We prove that a necessary and sufficient condition on the normalization condition that ensures commensuarability is the geometric mean-based normalization. Consequently, adopting an arithmetic mean-based normalization condition results in index values that depend on arbitrarily chosen measurement units, such as gallons or 100 milliliters.

Key Words: Cost of Living, Price Index, Preference Heterogeneity, Characterization

JEL Codes: E31, C43

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1 Introduction

The importance of considering changes in preferences when measuring inflation has been emphasized in recent studies, such as Hottman et al. (2016), Redding and Weinstein (2020), and Braun and Lein (2021). Indeed, it is unnatural to assume constant preferences when goods differ over time, due to seasonal fluctuations, or due to major shocks like the COVID-19 pandemic. Incorporating differences in preferences into inflation measurement is a natural idea. However, in the standard economic theory of price indices, the cost-of-living index a la Konüs (1939), preferences are assumed to be constant. Thus, changes in preferences must be integrated by modifying the Konus index. Recent research by Carluccio et al. (2023) (hereafter CGG) follows Hottman et al. (2016) in assuming cardinal utility functions and imposes normalization conditions on changing taste (appeal) parameters to compare prices under varying preferences. However, unlike Hottman et al. (2016) and Redding and Weinstein (2020), who impose normalization conditions to keep the geometric mean of preference parameters constant, CGG imposes a condition where the arithmetic mean of preference parameters remains constant. CGG justifies this with considerations from previous research. Redding and Weinstein (2020) concluded that the choice between arithmetic and geometric means does not make a big difference to the price index. However, recent critiques by Martin (2022) and Kurtzon (2020) suggest that the choice of normalization condition used by Redding and Weinstein (2020) has a significant impact on results. The problem is that it is difficult to decide from price and quantity information which normalization condition is appropriate. If the choice of normalization condition affects the results, it must be made carefully.

This paper shows that imposing the axiom of commensurability, the invariance of the price index from the choice of measurement units, which is fundamental in index number theory, leads to geometric mean-based normalization condition. The arithmetic mean-based normalization condition used by CGG means that the value of the price index depends on the choice of the measurement units of the goods used. For instance, the price index number changes depending on whether the price of orange juice is measured per liter or per gallon. Usually, the measurement units of products are freely determined by manufacturers or data creators without any economic theory. Therefore, it is undesirable for such choices of measurement units to influence the index number value. Independence from measurement units is considered a particularly important axiom in price index number theory, and almost all famous index number formula like the Laspeyres, Fisher, and Sato-Vartia indices meet this requirement. Moreover, Redding and Weinstein (2020) and Hottman et al. (2016) use a geometric mean-based normalization condition, which has independence from measurement units and does not pose a problem in this regard.

This paper first demonstrates with examples how the value of a price index depends on the choice of measurement units when an arithmetic mean-based normalization condition is imposed. It then shows that imposing commensurability property leads to a geometric mean-based normalization condition, and further, imposing equal treatment of each product uniquely derives the simple geometric mean-based normalization condition used by Hottman et al. (2016) and Redding and Weinstein (2020) as a necessary and sufficient condition.

2 Cost of Living Index under Variable Preferences

The Konüs (1939) cost of living index (COLI) is defined as the ratio of expenditure functions at two periods given same preferences in both periods. Following Redding and Weinstein (2020), we assume that the utility function is of the class of constant elasticity of substitution (CES) as follows;

$$U_t = \left(\sum_{i=1}^{N} \left(\varphi_i q_{it} \right)^{\frac{\sigma - 1}{\sigma}} \right)^{\frac{\sigma}{\sigma - 1}}, \tag{1}$$

where $\sigma > 0$ is the elasticity of substitution. q_{it} is the quantity of commodity i at time $t, \varphi_i \geq 0$ is a preference parameter that affects the marginal utility of commodity i. N is the number of different commodities.

The cost of living index for the CES preference is given by

$$COLI\left(s,t\right) = \frac{\left(\sum_{i=1}^{N} \left(\frac{p_{it}}{\varphi_i} \right)^{1-\sigma} \right)^{\frac{1}{1-\sigma}}}{\left(\sum_{i=1}^{N} \left(\frac{p_{is}}{\varphi_i} \right)^{1-\sigma} \right)^{\frac{1}{1-\sigma}}} \tag{2}$$

When the preference parameters vary over time, we add a subscript t to the preference parameter φ_i in Equation (1). The utility function becomes

$$U_t = \left(\sum_{i=1}^{N} (\varphi_{it} q_{it})^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}, \tag{3}$$

The cost of living index widely used in recent literature can be written as

$$COLI\,(s,t) = \frac{\left(\sum_{i=1}^{N} \left(\frac{p_{it}}{\varphi_{it}} \right)^{1-\sigma} \right)^{\frac{1}{1-\sigma}}}{\left(\sum_{i=1}^{N} \left(\frac{p_{is}}{\varphi_{i}s} \right)^{1-\sigma} \right)^{\frac{1}{1-\sigma}}} \tag{4}$$

Note that the right hand side of Equation (4) is not homogeneous of degree zero with respect to the taste parameters,φ_{it} and φ_{is}. That is, this COLI is not invariant to monotonic transformations of the utility function (3). In order to identify the COLI under variable preferences, (4), we need to impose an exogenous normalization condition for the taste parameters. Redding and Weinstein (2020) assume the following normalization condition

$$\frac{1}{N} \sum_{i=1}^{N} \ln \varphi_{it} = \ln \varphi \text{ for all } t.\ \varphi > 0. \tag{5}$$

That is, they impose that the geometric means of the preference parameters remain constant over time.

In their paper, CGG adopt the following normalization condition.[1]

$$\sum_{i=1}^{N} \varphi_{it} = 1 \text{ for all } t. \tag{6}$$

Note that from the first order condition for the cost minimization, we can obtain the following relations between the taste parameters, φ_{it}, and the expenditure share at time t, w_{it},

$$\frac{\varphi_{it}}{\varphi_{1t}} = \frac{p_{it}}{p_{i1}} (\frac{w_{it}}{w_{1t}})^{\frac{1}{\sigma-1}}. \tag{7}$$

Using the above relation as well as the normalization condition, we can solve φ_{it} as a function of the relative prices and expenditure shares, which gives us a general price index for bilateral comparisons. The actual derivation of the cost of living index is

[1] Due to differences in the specification of the utility function, the actual normalization condition in CGG includes σ. But, this does not affect the following arguments.

straightforward.

Denote the unit cost function as follows:

$$P_t = \left(\sum_{i=1}^{N} \left(\frac{p_{it}}{\varphi_{it}} \right)^{1-\sigma} \right)^{\frac{1}{1-\sigma}}.$$

Then, from the first order condition for cost minimization, we can obtain the following relation.

$$\ln P_t = -\ln \varphi_{it} + \ln p_{it} - \frac{1}{1-\sigma} \ln w_{it}, \tag{8}$$

Based on the above first order condition, using the normalization condition, it is easy to derive the cost of living index. For example, Redding and Weinstein (2020) derive the following price index number formula for their cost of living index,

$$\ln COLI = \frac{1}{N} \sum_{i=1}^{N} \left((\ln p_{it} - \ln p_{is}) - \frac{1}{(1-\sigma)} (\ln w_{it} - \ln w_{is}) \right). \tag{9}$$

If we use the arithmetic normalization condition adopted in Carluccio et al. (2023), we can obtain the following formula

$$COLI\,(s,t) = \frac{\sum_{i=1}^{N} p_{it} w_{it}^{(1/(\sigma-1))}}{\sum_{i=1}^{N} p_{is} w_{is}^{(1/(\sigma-1))}}. \tag{10}$$

The binary price index in (10) is very similar to the Dutot's index as follows:

$$Dutot\,(s,t) = \frac{\sum_{i=1}^{N} p_{it}}{\sum_{i=1}^{N} p_{is}} \tag{11}$$

The Dutot's index, (11), commonly used in various countries as the formula for elementary level indices, is limited to aggregating identical or very similar products. This limitation arises because the index violates the principle of commensurability, restricting its applicability. The index number (10) based on the arithmetic normalization condition in (6) also violates commensurability.

3 Commensurability and Normalization Conditions

A possible criticism against the COLI in (4) as a general bilateral price index is that it depends on the choice of the normalization of the taste parameters. Suppose the utility functions at time m, denoted by U_m, then under cardinal preferences the COLI under variable preferences can be written as

$$COLI = \frac{E_t\left(p_t\right) \times U_t}{E_s\left(p_s\right) \times U_s} \tag{12}$$

where $U_t = U_s = U$. $E_t(p_t)$ is the unit expenditure function which is the amount of expenditure per one unit of utility. It is clear that CCG assumes that the levels of different utility functions are comparable. To make the comparison possible, we need to impose structures on the preference parameters. More specifically, when preferences are CES over N commodities, there are N taste parameters. One of them cannot be identified from economic data on prices and quantities. However, to make price comparisons with different utility functions, all the taste parameters must be specified.[2] One natural assumption is to set one of the commodities as the base commodity and fix the taste parameter at a constant value, such as $\varphi_1 = 1$. Alternatively, we can assume adding up conditions such as $\sum_{i=1}^{N} \varphi_i = 1$. Actually, there are infinitely many possible normalization conditions. The main problem is that economic theory does not help us with the choice of a normalization condition. Redding and Weinstein (2020) considered the class of functions of quadratic means of order r as the potential set of the normalization conditions. They reported that their COLI is not sensitive to the choice of r. However, one may argue that some arbitrariness still remains in choosing the class of the functional forms of the normalization conditions. Another specification proposed by Kurtzon (2020) is a normalization condition using a weighted geometric average of taste parameters with expenditure share weights, which leads to the standard Sato-Vartia index without any taste shifts.

In this section, we characterize the class of normalization conditions on taste parameters for CCG that satisfy *commensurability test/axiom.* Commensurability is one of the most fundamental of axioms for price index number formula. Without commensurability,

[2] If the COLI is ordinal, we do not need to specify all the taste parameters so that the normalization condition is not necessary. However, in such a case, additional exogenous information such as the reference quantity vector is required. See Balk (1989) for details of ordinal COLI with heterogenous preferences.

choice of measurement unit, such as litre or gallon, kilogram or pound, can affect the resulting price index. Almost all the price index number formula to date pass the commensurability test. We show that, surprisingly, CCG passes the commensurability test only when the normalization takes the form of a geometric mean.

In this paper, following previous literature such as CCG, we consider a specific class of normalization conditions, defined as:

$$h\left(\varphi_{1m},,\varphi_{2m},...,,\varphi_{Nm}\right)=1 \text{ for } m=t,s. \tag{13}$$

This implies that the normalization condition is applied to the preference parameters for each time period separately. Furthermore, we assume that this normalization condition is invariant over time.

The taste parameter for commodity i at time m is expressed as,

$$\varphi_{im}=\left(\frac{p_{im}}{p_{1m}}\right)\left(\frac{w_{im}}{w_{m1}}\right)^{\frac{1}{\sigma-1}}\varphi_{1m}, \text{ for all } i=1,..,N, \text{ and } m=t,s. \tag{14}$$

That is, the taste parameter for each commodity i can be obtained from the relative prices and expenditure shares of commodities i and 1.

Given the functional relationship of taste parameters with price and expenditure shares, we modify the normalization condition (13) to:

$$\varphi_1=f\left(\varphi_2,...,\varphi_N\right). \tag{15}$$

We note that, due to the time invariance of the normalization condition, we omit the time subscripts m,t,s for simplicity. We can substitute (14) into (15) , which leads to

$$\varphi_1=f\left(x_{2m}\varphi_1,x_{3m}\varphi_1,...,x_{Nm}\varphi_1\right) \tag{16}$$

where $x_{im}=\left(\frac{p_{im}}{p_{1m}}\right)\left(\frac{w_{im}}{w_{1m}}\right)^{\frac{1}{\sigma-1}}$ for all *i=1,..,N,* and *m=t,s.*

Now, define the function, g, which solves the above for φ_1 such that

$$\varphi_1=g\left(x_{2m},x_{3m},...,x_{Nm}\right),$$

where $g:\mathbf{R}_{++}^{N-1}\rightarrow\mathbf{R}_{++}$ is a positive valued continuous function.

For example, if we use a geometric mean as the normalization condition:

$$\frac{1}{N}\sum_{i=1}^{N}\ln\varphi_{im}=\ln\varphi, \tag{17}$$

From the definition of x_{im}, it is easy to obtain the following relation,

$$\varphi_{1m}=g\left(x_{2m},x_{3m},...,x_{Nm}\right)=\varphi\prod_{i=2}^{N}x_{it}^{-1/N} \tag{18}$$

Similarly, if we use an arithmetic normalization condition as follows:

$$\sum_{i=1}^{N}\varphi_{im}=1, \tag{19}$$

the corresponding g function becomes

$$\varphi_{1}=g\left(x_{2m},x_{3m},...,x_{Nm}\right)=\frac{1}{1+\sum_{i=2}^{N}x_{im}}. \tag{20}$$

Proposition *Commesurability and Normalization*

Suppose the normalization condition takes the form of $\varphi_1=g\left(x_{t2},x_{t3},...,x_{tN}\right)$ *where* $g:\mathbf{R}_{++}^{N-1}\rightarrow\mathbf{R}_{++}$ *is a continuous function with* $x_{it}=\left(\frac{p_{it}}{p_{1t}}\right)\left(\frac{w_{it}}{w_{1t}}\right)^{\frac{1}{\sigma-1}}$ *for* $i=1,..,N.$ *Then, the COLI defined in* (4) *passes the commensurability test if and only if the normalization condition can be written as*

$$\varphi_1=A\left(\prod_{i=2}^{N}x_{ii}^{c_i}\right)$$

where $A>0,c_i\geq 0.$

Proof

From (8), the COLI under variable preferences can be written as

$$\ln COLI=\ln P_t-\ln P_s=-(\ln\varphi_{1t}-\ln\varphi_{1s})+(\ln p_{1t}-\ln p_{1s})-\frac{1}{1-\sigma}(\ln w_{1t}-\ln w_{1s}) \tag{21}$$

Because the first and second terms of the R.H.S. of (21) are obviously free from the measurement unit, the necessary and sufficient condition for the COLI to satisify the commensurabililty is to have the first term of the R.H.S. of (21) be free from the measurement unit. Therefore, it suffices for us to show the condition such that the term,

$(\ln \varphi_{1t} - \ln \varphi_{1s})$, is commensurable.

First, we prove the result when $N = 2$.

From the definition of the taste parameter, we have.

$$\varphi_{2t} = \left(\frac{p_{2t}}{p_{1t}}\right)\left(\frac{w_{2t}}{w_{1t}}\right)^{\frac{1}{\sigma-1}} \varphi_{1t}.$$

Suppose a change in measurement units occurs, so that we get a new price, p^*_{it}, and quantity, q^*_{it}, as follows,

$$p^*_{it} = p_{it}\lambda_i,$$
$$q^*_{it} = q_{it}/\lambda_i$$
$$\text{where } \lambda_i > 0 \text{ for } i = 1 \text{ or } 2.$$

Also denote φ^*_{t1} as the taste parameter for commodity i after the change in the measurement units. Then as is shown before, the necessary and sufficient condition for the commensurability is as follows:

$$\ln \varphi_{1s} - \ln \varphi^*_{1s} = \ln \varphi_{1t} - \ln \varphi^*_{1t}.$$

Denote the difference of the taste parameter as $\ln K$, that is,

$$\ln \varphi_{1s} - \ln \varphi^*_{1s} = \ln \varphi_{1t} - \ln \varphi^*_{1t} = \ln K \tag{22}$$

Also denote

$$x_t = \left(\frac{p_{2t}}{p_{1t}}\right)\left(\frac{w_{2t}}{w_{1t}}\right)^{\frac{1}{\sigma-1}}$$
$$x^*_t = \left(\frac{p^*_{t2}}{p_{t1}}\right)\left(\frac{w_{t2}}{w_{t1}}\right)^{\frac{1}{\sigma-1}}$$
$$\lambda = \left(\frac{\lambda^*_2}{\lambda_1}\right).$$

By assumption, the normalization condition is

$$\varphi_{t1} = g\left(x_t\right).$$

Then, after the change in the measurement units, we have

$$\begin{aligned}\varphi_{t1} &= g\left(x_t^*\right) \\ &= g\left(\lambda x_t\right).\end{aligned}$$

Note that (22) can be written as

$$g\left(\lambda x_t\right) - g\left(x_t\right) = g\left(\lambda x_s\right) - g\left(x_s\right) = \ln K.$$

The above equation must hold for any values for x_t and x_s. Therefore, while K depends on λ, K does not depend on x_t nor x_s.

If the index passes the commensurability test, given λ, there is a constant $K\left(\lambda\right) > 0$ for all x_t that satisfies the following equations.

$$g\left(x_t\right) = \frac{1}{K\left(\lambda\right)} g\left(\lambda x_t\right).$$

Then, we have

$$g\left(\lambda x_t\right) = g\left(x_t\right) \times K\left(\lambda\right) \tag{23}$$

Let $f(\lambda x_t) = g(\lambda x_t)$, then, we get

$$f(\lambda x_t) = g\left(x_t\right) \times K\left(\lambda\right) \tag{24}$$

This is a well known functional equation whose general solution[3] for $x_t > 0$, and $\lambda > 0$ is given by

$$f(x_t) = g(x_t) = a \times x_t^c \tag{25}$$

$$K(\lambda) = b \times \lambda^c \tag{26}$$

where $a, b, c \in \mathbf{R}$.

Therefore, to make the COLI commensurable, the normalization condition must be of the following form,

[3]See Aczél (1966) and Eichhorn (1978) for the derivations of the general solution.

$$\varphi_{1t} = Ax_{2t}^{c}.$$

Then, consider the case with $N \geq 3$. For any $p_t, p_s, q_t, q_s \in \mathbf{R}_{++}^N$, consider a potivae valued vector, $\lambda_i \in \mathbf{R}_{++}$. Then the new price and quantity vectors after change in units of measurement are as follows,

$$\begin{aligned}
p_t^* &= (\lambda_1 p_{1t}, \lambda_2 p_{2t}, ..., \lambda_N p_{Nt}), \\
p_s^* &= (\lambda_1 p_{1s}, \lambda_2 p_{2s}, ..., \lambda_N p_{Ns}), \\
q_t^* &= \left(\frac{q_{1t}}{\lambda_1}, \frac{q_{2t}}{\lambda_2}, ..., \frac{q_{Nt}}{\lambda_N} \right), \\
q_s^* &= \left(\frac{q_{1s}}{\lambda_1}, \frac{q_{2s}}{\lambda_2}, ..., \frac{q_{Ns}}{\lambda_N} \right).
\end{aligned}$$

Suppose we have the following normalization condition,

$$\varphi_1 = g(\varphi_2, ..., \varphi_N).$$

Using the definition of the taste parameters, the normalization condition can be written as

$$\varphi_1 = g(x_{2t}, x_{3t}, ..., x_{Nt}).$$

where

$$x_{it} = \left(\frac{p_{it}}{p_{1t}} \right) \left(\frac{w_{it}}{w_{1t}} \right)^{\frac{1}{\sigma - 1}}.$$

After the change in the measurement units, we have

$$\begin{aligned}
\varphi_{1t}^* &= g(x_{2t}^*, x_{3t}^*, ..., x_{Nt}^*) \\
&= g(\pi_2 x_{2t}, \pi_3 x_{3t}, ..., \pi_N x_{tN})
\end{aligned}$$

where

$$\pi_i = \left(\frac{\lambda_i}{\lambda_1} \right).$$

As shown before, the necessary and sufficient condition for commensurability is

$$\ln \varphi_{1s}^* - \ln \varphi_{1s} = \ln \varphi_{1t}^* - \ln \varphi_{1t}.$$

Therefore,

$$\begin{aligned}&\ln g\left(x_{2t}^*, x_{3t}^*, ..., x_{Nt}^*\right) - \ln g\left(x_{2t}, x_{3t}, ..., x_{Nt}\right)\\&= \ln g\left(x_{2s}^*, x_{3s}^*, ..., x_{Ns}^*\right) - \ln g\left(x_{2s}, x_{3s}, ..., x_{Ns}\right).\end{aligned}$$

Define a vector, π as

$$\pi = \left(\pi_2, \pi_3, ..., \pi_N\right).$$

Then, we can rewrite the above conditions as

$$\begin{aligned}&= \ln g\left(\pi_2 x_{2s}, \pi_3 x_{3s}, ..., \pi_N x_{Nss}\right) - \ln g\left(x_{s2}, x_{s3}, ..., x_{sN}\right)\\&= \ln g\left(\pi_2 x_{2t}, \pi_3 x_{3t}, ..., \pi_N x_{Nt}\right) - \ln g\left(x_{t2}, x_{t3}, ..., x_{tN}\right)\\&\equiv K.\end{aligned}$$

Since these conditions must hold for any x_{ti} and x_{si}, K is independent from x_{ti} and x_{si}, but a function of π.Therefore, it is possible to rewrite the necessary and sufficient condition as

$$\begin{aligned}\varphi_{t1} &= \frac{1}{K\left(\pi\right)} g\left(\pi_2 x_{2t}, \pi_3 x_{3t}, ..., \pi_N x_{Nt}\right)\\&= g\left(x_{2t}, x_{3t}, ..., x_{Nt}\right)\end{aligned}$$

Therefore, we get the following necessary and sufficient condition

$$g\left(\pi_2 x_{2t}, \pi_3 x_{3t}, ..., \pi_N x_{Nt}\right) = K\left(\pi\right) \times g\left(x_{2t}, x_{3t}, ..., x_{Nt}\right) \tag{27}$$

From Theorem 1 in Luce (1964), we can show that (27) has the following general solution,

$$g\left(x_{2t}, x_{3t}, ., x_{Nt}\right) = A(\prod_{i=2}^{N} x_{it}^{c_i}) \tag{28}$$

(End of Proof)

Note that the above normalization condition includes $\prod_{i=1}^{N} \varphi_{it} = 1$ as a special case, but not $\sum_{i=1}^{N} \varphi_{it} = 1$. Therefore, if we use arithmetic mean as the normalization condition, the COLI becomes dependent on the choice of the measurement units of commodities. More general normalization such as the quadratic mean of order r also fails the commensurability test. In a recent paper, Kurtzon (2020) discusses normalization conditions involving expenditure share weighted average of taste parameters using observed expenditure shares at times s or t. The resulting index is identical to the the Sato-Vartia index which is commensurable. Our approach differs from Kurtzon (2020) for two key reasons: Firstly, the taste (appeal) parameters are considered as exogenous structural parameters, unaffected by market activities. Secondly, expenditure shares are functions of these taste parameters, necessitating that conditions on taste parameters remain independent of expenditure shares.

If we restrict the normalization condition so that all the taste parameters, φ_{it}, are considered equally important and treated symmetrically, the necessary and sufficient condition can be written in the form of a simple geometric mean, which is identical to the normalization condition by Redding and Weinstein (2020). Formally, we can derive the corollary as follows.

Corollary *Unweighted Geometric Mean*
In addition to the assumptions in the Proposition above, if we impose a restriction such that all the taste variables, φ_{it}, are treated equally, the necessary and sufficient condition can be written as follows:

$$\frac{1}{N}\sum_{i=1}^{N} \ln \varphi_{it} = \ln \varphi. \tag{29}$$

Proof

Because all the parameters are treated equally, we get

$$c_i = \beta \textbf{ for all } i = 2, 3, , N.$$

Then, Equation (28) becomes

$$\varphi_{1t} = A(\prod_{i=2}^{N} x_{it}^{\beta})$$

By the definition of x_{it}, we can obtain

$$\varphi_{1t} = A(\prod_{i=2}^{N} \varphi_{it}^{\beta}) \times \varphi_{1t}$$

Then, find β that satisfies the following equation:

$$\beta = -(N-1)\beta - 1$$

The solution is

$$\widehat{\beta} = \frac{-1}{N}.$$

Using $\widehat{\beta}$, and setting

$$\varphi = A^{\frac{-1}{\beta}},$$

we get

$$\frac{1}{N}\sum_{i=1}^{N} \ln \varphi_{it} = \ln \varphi.$$

(End of Proof)

The following two tables contains numerical examples that show the commensurability properrty associated with different normalizations. We assume there are three goods whose prices and quantities change between time 1 an 2. The elasticity of substitution, σ, is set at 3.

Time	Commodiy	Price	Quantity
1	1	1000	2
	2	200	20
	3	150	30
2	1	500	3
	2	100	30
	3	20	40

Table 1: Case 1

In Table 2, we change the measurement unit of the first product so that the price becomes 1/10 while the quantity becomes 10 times greater than in Case 1.

Time	Commodiy	Price	Quantity
1	1	100	20
	2	200	20
	3	150	30
2	1	50	30
	2	100	30
	3	20	40

Table 2: Case 2

Table 3 reports the price index number values between time 1 and 2 for Case 1 and Case 2.

	Sato-Vartia	Geometric Case	Arithmetic Case
Case 1	0.3476	0.3086	0.5303
Case 2	0.3476	0.3086	0.4132

Table 3: Price Index Number Values

While the Sato-Vartia index and the COLI with the geometric mean, (5) are identical between the two cases, the CGG COLI with the arithmetic condition, (6), changes from 0.5303 to 0.4132. This example illustrates the dependence of the COLI with the arithmetic normalization condition such as (6) on the choice of the measurement unit.

The intuition of the failure of commensurability is as follows. As is clear from (14), a change in taste parameters affects the value of the COLI unless the taste effects at time t and s are cancelled out each other. By definition, the ratio of two taste parameters is closely related with the relative price among commodities as is clear from the following equation.,

$$\frac{\varphi_{im}}{\varphi_{1b}} = \left(\frac{p_{im}}{p_{m1}}\right)\left(\frac{w_{im}}{w_{1m}}\right)^{\frac{1}{\sigma-1}}.$$

Suppose the measurement unit for commodity i is changed so that p_{im} is multiplied by λ, that is, we have $p^*_{im} = \lambda p_{im}$. Without normalization condition, such a change causes a proportional change in φ_{im} so that the new taste parameter, φ^*_{im}, is equal to $\lambda\varphi_{im}$. However, the normalization condition makes other taste parameters, $\varphi_{jm}\,(j \neq i)$ vary. Suppose the change in the sum of the taste effects at time t caused by the change in the measurement unit of commodity i is always the same as the sum of effects at time s. This implies that the change in the taste parameter, $\varphi^*_{jm}/\varphi_{jm}$, should be independent from the level of φ_{jm}, so that we can write $\varphi^*_{jm}/\varphi_{jm} = h(\lambda)$. Or, we must have $\ln\varphi^*_{jm} = \ln h(\lambda) + \ln\varphi_{jm}$. This implies that the effects caused by a change in the measurement unit of commodity i, affects other taste parameters multiplicatively, which restricts the normalization condition to a class of multiplicative functions over taste parameters.

4 Conclusions

In this paper, we show that the cost of living index with variable preferences that has been widely used depends crucially on the choice of the normalization condition. As a price index number formula, the price index must be independent from the choice of the measurement unit of commodities. We have provided a necessary and sufficient condition for the normalization condition to ensure commensurability. The paper has also shown that, under a symmetric treatment of taste parameters, the only normalization that leads to a price index which satisfies commensurability is the equally weighted specification used in Redding and Weinstein (2020) and Hottman et al. (2016).

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