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Vintage Depreciation, User Costs, and Total Factor Productivity:

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Abstract

We develop a vintage-accounts framework for real estate capital measurement. The *Mutual Determination Lemma* formalises the equivalence among vintage prices, user costs, and depreciation rates—a constraint routinely violated in applied work. The *Maintenance-Depreciation Theorem* shows that linearly increasing maintenance costs produce a strictly accelerating depreciation profile, rationalising geometric-type patterns. The *Geometric PIM Decomposition Theorem* proves that geometrically distributed retirement ages are necessary and sufficient for a geometric population PIM, establishing Diewert (2004)’s $\delta^* = \delta^{CS} + \delta^{demolish}$. The *Land-Separation Bias Theorem* derives signed first-order and non-negative second-order expressions for the TFP bias from conflating land and structure. Illustrative simulations quantify each result.

JEL codes: C43, D24, D92, E22, R33

Keywords: Vintage capital; depreciation; user cost; TFP; Builder’s Model; land–structure decomposition

*This paper originates in and is deeply indebted to the capital measurement theory of W. Erwin Diewert—in particular his “Measuring Capital” (Discussion Paper 04-10, University of British Columbia, 2004)—which established the vintage-accounts duality that underlies all four theorems of the present paper. It is presented on the occasion of Professor Diewert’s award, in recognition of his fundamental contributions to index number theory and capital measurement. Financial support from JSPS KAKENHI Grant 24H00012 is gratefully acknowledged. All errors are the author’s own.

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1 Introduction

Measuring the contribution of capital to production requires, as a first step, an accurate model of how productive capacity declines with asset age. For real property this step is particularly fraught: the observable price of a commercial property conflates a depreciating structure with a permanent plot of land, and five competing models of structural depreciation — one-hoss-shay, straight-line, geometric, age-dependent geometric, and maintenance-expenditure — imply markedly different capital-service flows from physically identical assets.

This paper resolves the resulting measurement ambiguity within a unified vintage-accounts framework. Four results are established as theorems.

Contribution 1 (Identification). The *Vintage-Accounts Identification Lemma* (Theorem 2.4) provides a self-contained algebraic formalisation of the mutual determination among vintage asset prices $\{P_n^t\}$, age-efficiency profiles $\{f_n^t\}$, and cross-section depreciation rates $\{\delta_n^t\}$. The substance of this equivalence was first stated explicitly by [Jorgenson and Griliches \(1967\)](#) and formalised by [Jorgenson \(1989\)](#) and [Hulten \(1990\)](#); it is central to the [Diewert \(2004\)](#) vintage-accounts framework. Our contribution is not new substance but new form: a model-free algebraic proof, stated as a theorem, that makes explicit a constraint routinely violated in applied capital-measurement work. Specifically, part (iv) shows that independently specifying both a depreciation profile and an efficiency profile over-identifies the system—a diagnostic implication that [Diewert \(2004\)](#) noted informally but did not formalise.

Contribution 2 (Maintenance and Accelerated Depreciation). The *Maintenance-Depreciation Theorem* (Theorem 3.8) characterises the net depreciation profile of a building facing linearly increasing maintenance costs. [Diewert \(2004\)](#) observed in passing that increasing maintenance costs cause used asset prices to exhibit “the profile implied by geometric depreciation (up to some limiting age)” (Ch. 1, Appendix A5) but did not formalise this observation. Theorem 3.8 provides the explicit characterisation: the cross-section depreciation

rate is $\delta_n^E = d/(1 - nd) + O(r^*)$, strictly increasing in age n —not constant, as exact geometric would require. The theorem therefore formalises Diewert’s observation while being precise about its scope: the maintenance model rationalises *accelerated* depreciation (consistent with [Hulten and Wykoff 1981](#) and [Diewert and Shimizu 2017](#)), not exact geometric. The profile transitions continuously from one-hoss-shay ($d \rightarrow 0$) through accelerating (Technology E) to straight-line efficiency decline ($d \rightarrow 1/L^*$), and the theorem yields exact comparative statics: $\partial L^*/\partial d < 0$ and $\partial L^*/\partial g > 0$.

Contribution 3 (Geometric PIM Decomposition). The *Geometric PIM Decomposition Theorem* (Theorem 4.6) is the central result of this paper. It proves that when (i) each building’s net service flow follows geometric efficiency decline and (ii) optimal retirement ages are geometrically distributed with hazard rate λ , the *population* capital-services aggregate satisfies an exact geometric PIM with rate $\delta^* = \delta^{CS} + \lambda - \delta^{CS}\lambda$.

Three aspects are new relative to the existing literature.

The decomposition proof. [Diewert \(2004\)](#) proposed $\delta^* = \delta^{CS} + \delta^{demolish}$ and calibrated it numerically, but did not prove it. In [Diewert \(2004\)](#), Ch. 6 established only the *sufficiency* direction: geometric individual efficiency implies geometric population PIM. Theorem 4.6 proves the decomposition exactly and establishes the *equivalence* (Part iv): geometric retirement-age distribution is *necessary and sufficient* for the population PIM to be geometric. The necessity direction is new.

The survivors-bias corollary. The survivors-bias problem has a long history. [Schmalenbach \(1919\)](#); cited in [Diewert 2004](#), Ch. 1) warned: “those which one sees are but the rare survivors; the many dead have long lain buried—this can be the source of serious errors.” [Hulten and Wykoff \(1981\)](#) and [Hulten \(1990\)](#) noted that cross-section estimators must be adjusted for early retirement of assets, but neither provided a theoretical characterisation of the direction of the resulting bias. Part (v) of Theorem 4.6 provides the first proof that cross-section estimators systematically *understate* δ^* by $\delta^{demolish} \approx \lambda = 1/\bar{L}$, and identifies

the heterogeneity in d_i as the mechanism.

Quantitative significance. In Japan, where $\bar{L} \approx 30$ years, ignoring $\delta^{demolish}$ understates total depreciation by 61% and overstates TFP by approximately $\bar{s}_S \times \delta^{demolish} \approx 0.23\%/yr$ (see Table 1 and Section 7).

Contribution 4 (Land-Separation Bias). The *Land-Separation Bias Theorem* (Theorem 6.1) embeds the Builder’s Model of Diewert and Shimizu (2017) into the vintage-accounts framework and derives a closed-form expression for the TFP bias that arises from treating land and structure as a single undepreciated asset. The bias is proportional to the land-value share and the land–structure price-growth differential. Using Tokyo REIT sample statistics from Diewert and Shimizu (2017) ($\bar{s}_K \approx 0.80$, $\bar{\phi}_S \approx 0.07$, $\bar{\phi}_L \approx 0.93$), a land–structure price-growth differential of $\mu^t = 5\%/yr$ (consistent with the Tokyo bubble period) generates a first-order bias $B^t \approx 0.80 \times 0.07 \times 0.93 \times 0.05 \approx 0.26\%/yr$, a non-trivial fraction of typical TFP growth estimates.

Illustrative simulations. Section 7 quantifies each theorem’s implications through simulations calibrated to published Tokyo REIT estimates (Diewert and Shimizu, 2017). No new estimation is performed.

Related literature. The vintage capital and user-cost framework originates with Walras (1874), Böhm-Bawerk (1891), and Hotelling (1925), and is synthesised by Christensen and Jorgenson (1969), Hulten and Wykoff (1981), Jorgenson (1989), Hulten (1990), and Diewert (2004). The measurement of capital in national accounts is covered by Schreyer (2001) and Schreyer (2009). The Builder’s Model and its commercial-property applications appear in Diewert and Shimizu (2015), Diewert and Shimizu (2017), and Diewert and Shimizu (2016b); the SNA linkage is developed in Diewert, Fox and Shimizu (2016) and Diewert, Nomura and Shimizu (2024). The land–structure separation in productivity accounts is emphasised by Schreyer and Diewert (2014). Our contributions to these literatures are: a model-free for-

malisation of the vintage-accounts identification constraint; a precise characterisation of how maintenance costs drive accelerated depreciation; a proof of the geometric-PIM decomposition with its equivalence and survivors-bias corollaries; and a closed-form land-separation bias expression for TFP.

Organisation. Section 2 presents the vintage-accounts framework and proves Theorem 2.4. Section 3 characterises the five depreciation technologies and proves Theorem 3.8. Section 5 derives user-cost formulae for land and structure. Section 6 proves Theorem 6.1. Section 7 presents illustrative simulations. Section 8 concludes. All proofs are in Appendix A.

2 The Vintage-Accounts Framework

2.1 Primitives

Let $t \in \{1, \dots, T\} \subset \mathbb{N}$ denote an accounting period and $n \in \{0, 1, \dots\}$ the age of a capital asset in periods. Throughout we study a single homogeneous class of reproducible capital with finite maximum life $L \in \mathbb{N}$.

Definition 2.1 (Vintage objects). Fix period t and age $n \in \{0, \dots, L\}$.

- (i) *Vintage asset price:* $P_n^t \in \mathbb{R}_{++}$ for $n < L$ and $P_L^t = 0$. P_n^t is the spot price, at the *beginning* of period t , of a unit aged n . The finite sequence $\mathbf{P}^t \equiv (P_0^t, \dots, P_L^t)$ is the *vintage price profile* at time t .
- (ii) *Beginning-of-period user cost:* $f_n^t \geq 0$. f_n^t is the rental paid at the *start* of period t for use of a unit aged n during period t . The sequence $\mathbf{f}^t \equiv (f_0^t, \dots, f_{L-1}^t)$ is the *age-efficiency profile*.
- (iii) *End-of-period user cost:* $u_n^t \equiv (1 + r^t)f_n^t$, where $r^t > 0$ is the one-period nominal interest rate at time t .

(iv) *Cross-section depreciation rate:*

$$\delta_n^t \equiv 1 - \frac{P_{n+1}^t}{P_n^t} \in [0, 1), \quad n = 0, \dots, L-1, \quad (2.1)$$

with the convention $\delta_{L-1}^t = 1$. The vector $\boldsymbol{\delta}^t \equiv (\delta_0^t, \dots, \delta_{L-1}^t)$ is the *cross-section depreciation profile*.

The sequences \mathbf{P}^t , \mathbf{f}^t , $\boldsymbol{\delta}^t$ are *cross-sectional* objects: they describe the state of the market for assets of different ages at a single point in time t . They are not the time-path of a single asset.

Assumption 2.2 (Flat term structure). For each t , the one-period nominal interest rate and the expected one-period rental-price escalation factor are independent of vintage age: $r_n^t = r^t \in \mathbb{R}_{++}$ and $i_n^t = i^t \in (-1, r^t)$ for all $n \geq 1$.

Under Assumption 2.2, define the *net discount factor*

$$\gamma^t \equiv \frac{1 + i^t}{1 + r^t} \in (0, 1). \quad (2.2)$$

The strict inequality $\gamma^t < 1$ follows from $i^t < r^t$.

2.2 The fundamental recursion

Lemma 2.3 (Price–efficiency recursion). *Under Assumption 2.2, for every t and $n \in \{0, \dots, L-1\}$,*

$$P_n^t = f_n^t + \gamma^t P_{n+1}^t. \quad (2.3)$$

Proof. At the beginning of period t , the owner of a unit aged n receives rental f_n^t and retains an asset worth P_{n+1}^t at the beginning of period $t+1$. The discounted return on investment must equal the purchase price: $f_n^t + [(1 + i^t)/(1 + r^t)]P_{n+1}^t = P_n^t$. ■

Solving (2.3) forward with $P_L^t = 0$ yields the *fundamental valuation equation*:

$$P_n^t = \sum_{k=0}^{L-1-n} (\gamma^t)^k f_{n+k}^t, \quad n = 0, \dots, L-1. \quad (2.4)$$

Substituting (2.1) into (2.3) and rearranging gives the *Walras–Jorgenson user-cost formula*:

$$u_n^t = [r^t - i^t + (1 + i^t) \delta_n^t] P_n^t. \quad (2.5)$$

2.3 The Vintage-Accounts Identification Theorem

Theorem 2.4 (Vintage-Accounts Mutual Determination). *Fix t and let (r^t, i^t) satisfy Assumption 2.2, so that $\gamma^t \in (0, 1)$. The following hold.*

(i) $\mathbf{f}^t \Rightarrow (\mathbf{P}^t, \boldsymbol{\delta}^t)$: *Given $\mathbf{f}^t \in \mathbb{R}_{\geq 0}^L$ with $f_0^t > 0$, the profile \mathbf{P}^t is uniquely determined by (2.4). Each $P_n^t > 0$ for $n < L$, and $\boldsymbol{\delta}^t$ follows from (2.1).*

(ii) $(\boldsymbol{\delta}^t, P_0^t) \Rightarrow (\mathbf{P}^t, \mathbf{f}^t)$: *Given $\boldsymbol{\delta}^t \in [0, 1]^{L-1} \times \{1\}$ and $P_0^t > 0$, the profile \mathbf{P}^t is uniquely determined by*

$$P_n^t = P_0^t \prod_{k=0}^{n-1} (1 - \delta_k^t), \quad n = 1, \dots, L, \quad (2.6)$$

and \mathbf{f}^t follows from inverting (2.3).

(iii) $\mathbf{P}^t \Rightarrow (\mathbf{f}^t, \boldsymbol{\delta}^t)$: *Given \mathbf{P}^t with $P_n^t > 0$ for $n < L$ and $P_L^t = 0$, both \mathbf{f}^t and $\boldsymbol{\delta}^t$ are uniquely determined by (2.1) and (2.3).*

(iv) Mutual determination: *The profiles $\mathbf{P}^t, \mathbf{f}^t, \boldsymbol{\delta}^t$ are mutually determined: no two of them can be specified independently while remaining consistent with (2.3).*

Proof. See Appendix A. ■

Remark 2.5 (Relationship to existing literature). The mutual determination of vintage prices, user costs, and depreciation rates dates at least to [Jorgenson and Griliches \(1967\)](#), who wrote that “an almost universal conceptual error in the measurement of capital input is

to confuse the aggregation of capital stock with the aggregation of capital service” (Jorgenson and Griliches, 1967, p. 257). The general framework was set out by Jorgenson (1989) and Hulten (1990), and is central to Diewert (2004). Diewert (2004) stated: “*given any one of these sequences or profiles, all of the other sequences are completely determined . . . assumptions about depreciation rates, the pattern of user costs by age or the pattern of asset prices by age cannot be made independently of each other*” (Ch. 1; see the discussion following eq. (0115) in that source). Theorem 2.4 elevates this observation to a self-contained, model-free algebraic theorem. Its practical value is part (iv): a researcher who independently specifies both a depreciation profile δ^t and an efficiency profile \mathbf{f}^t over-identifies the model—a constraint not always recognised in hedonic property-price studies that freely mix depreciation and age-efficiency assumptions.

2.4 Cross-section and time-series depreciation

The cross-section depreciation rate δ_n^t measures the *value loss from aging* across contemporaneous assets. National-accounts capital consumption corresponds to a distinct concept: the loss in the value of a *given* asset as one period elapses.

Definition 2.6 (Ex-ante time-series depreciation rate). The ex-ante time-series depreciation rate for an asset of age n in period t is

$$\pi_n^t \equiv 1 - (1 + i^{*t})(1 - \delta_n^t), \quad (2.7)$$

where $1 + i^{*t} \equiv (1 + i^t)/(1 + \rho^t)$, and ρ^t is the general price inflation rate.

Proposition 2.7 (Signed relationship between CS and TS depreciation). *Let $\delta_n^t \in (0, 1)$.*

- (i) $\pi_n^t = \delta_n^t$ if and only if $i^{*t} = 0$.
- (ii) $\pi_n^t < \delta_n^t$ if and only if $i^{*t} > 0$ (expected real capital gain).
- (iii) $\pi_n^t > \delta_n^t$ if and only if $i^{*t} < 0$ (expected real capital loss; obsolescence).

Proof. From (2.7): $\pi_n^t - \delta_n^t = -i^{*t}(1 - \delta_n^t)$. Since $1 - \delta_n^t > 0$, the sign of $\pi_n^t - \delta_n^t$ is $-\text{sgn}(i^{*t})$. ■

Remark 2.8. During a real estate boom ($i^{*t} > 0$), cross-section depreciation *overstates* time-series depreciation. Using δ_n^t in the TFP user-cost formula then *overstates* capital consumption, understates net capital input, and therefore *understates* TFP. This provides a theoretical explanation for the empirical pattern that commercial-property TFP appears to decline during asset-price booms.

2.5 Capital aggregates

Denote by $I^s > 0$ the gross real investment in period s . The vintage capital stocks available in period t are $K_n^t \equiv I^{t-n-1}$ for $n = 0, \dots, L-1$.

Definition 2.9 (Wealth stock and capital-services value).

$$W^t \equiv \sum_{n=0}^{L-1} P_n^t K_n^t, \quad \mathcal{S}^t \equiv \sum_{n=0}^{L-1} u_n^t K_n^t. \quad (2.8)$$

W^t is the *balance-sheet (wealth) capital stock*; \mathcal{S}^t is the *capital-services flow*.

Fisher chained indexes decompose the value ratios W^t/W^{t-1} and $\mathcal{S}^t/\mathcal{S}^{t-1}$ into price and quantity components whenever vintage prices are not proportional. Section 3.3 establishes a case in which proportionality holds exactly, eliminating the need for index-number aggregation.

3 Five Depreciation Technologies

For each technology we state the assumption, derive the implied profiles \mathbf{P}^t and \mathbf{f}^t , and give the form of the capital-services aggregate \mathcal{S}^t .

3.1 Technology A: One-hoss-shay

Assumption 3.1. $f_n^t = f_0^t$ for all $n \in \{0, \dots, L-1\}$; $f_n^t = 0$ for $n \geq L$.

By (2.4), the vintage price profile is

$$P_n^t = f_0^t \cdot \frac{1 - (\gamma^t)^{L-n}}{1 - \gamma^t}, \quad n = 0, \dots, L - 1. \quad (3.1)$$

Differencing: $D_n^t \equiv P_n^t - P_{n+1}^t = f_0^t (\gamma^t)^{L-n-1} / (1 - \gamma^t) > 0$, so $\delta_n^t = (\gamma^t)^{L-n-1} (1 - \gamma^t) / [1 - (\gamma^t)^{L-n}]$, which is strictly increasing in n : an asset delivering constant services depreciates faster in value terms as it ages.

3.2 Technology B: Straight-line depreciation

Assumption 3.2. $\delta_n^t = 1/(L - n)$ for $n = 0, \dots, L - 1$.

From (2.6): $P_n^t = (1 - n/L)P_0^t$ for all n .

3.3 Technology C: Geometric (constant-rate) depreciation

Assumption 3.3. $\delta_n^t = \delta \in (0, 1)$ for all n and t .

Proposition 3.4 (Aggregation simplification under geometric depreciation). *Under Assumption 3.3:*

(i) $P_n^t = (1 - \delta)^n P_0^t$ and $u_n^t = (1 - \delta)^n u_0^t$ for all n .

(ii) The capital-services flow satisfies $\mathcal{S}^t = u_0^t \cdot K_A^t$, where the productive capital stock K_A^t obeys the perpetual-inventory recursion

$$K_A^{t+1} = I^t + (1 - \delta)K_A^t. \quad (3.2)$$

(iii) No index-number formula is needed to aggregate over vintages: $P^F(\mathbf{P}^{t-1}, \mathbf{P}^t, \mathbf{K}^{t-1}, \mathbf{K}^t) = P_0^t$ and $Q^F(\cdot) = K_A^t$ for both W^t and \mathcal{S}^t .

Proof. (i): Induction on n from (2.6) and (2.5). (ii): Factor u_0^t from \mathcal{S}^t using (i). (iii): Hicks' Aggregation Theorem applies because all vintage prices are proportional. ■

The Diewert double-declining-balance conversion gives $\delta = 2/(L + 1)$ when the straight-line and geometric long-run stocks are equated under steady-state investment.¹

3.4 Technology D: Age-dependent geometric depreciation

Assumption 3.5. There exist breakpoints $0 = A_0 < A_1 < A_2 < \infty$ and rates $0 < \delta_1 \leq \delta_2 \leq \delta_3 < 1$ such that $\delta_n^t = \delta_k$ for $n \in [A_{k-1}, A_k)$, $k = 1, 2, 3$.

The vintage price profile is piecewise geometric: $P_n^t = P_0^t \cdot g(n)$, where

$$g(n) \equiv \begin{cases} (1 - \delta_1)^n & 0 \leq n < A_1, \\ (1 - \delta_1)^{A_1} (1 - \delta_2)^{n - A_1} & A_1 \leq n < A_2, \\ (1 - \delta_1)^{A_1} (1 - \delta_2)^{A_2 - A_1} (1 - \delta_3)^{n - A_2} & n \geq A_2. \end{cases} \quad (3.3)$$

Since vintage prices are not globally proportional, Fisher index aggregation over the three age cohorts is required.

Diewert and Shimizu (2017) estimate for Tokyo commercial offices: $(A_1, A_2) = (80, 120)$ quarters; $(\hat{\delta}_1, \hat{\delta}_2, \hat{\delta}_3) = (0.00327, 0.00702, 0.03558)$. Note $\hat{\delta}_3/\hat{\delta}_1 = 10.9$: buildings aged over 30 years depreciate at nearly eleven times the rate of new buildings.

3.5 Technology E: Linearly increasing maintenance expenditures

Technology E endogenises the retirement decision. Denote by g_n^t the *gross* service of an n -period-old unit (before maintenance) and by m_n^t the maintenance cost. The *net* service is $f_n^t = g_n^t - m_n^t$.

Assumption 3.6. (i) *Gross services are constant:* $g_n^t = \lambda^t g$ for all n , with $g > 0$ ($\lambda^t > 0$ is a period- t price-level factor).

(ii) *Maintenance costs increase linearly in age:* $m_n^t = \lambda^t (b + nc)$, with $g > b > 0$ and $c > 0$.

¹Under constant investment at unit rate, the straight-line stock is $(L + 1)/2$ and the geometric stock is $1/\delta$; equating gives $\delta = 2/(L + 1)$.

(iii) *Common escalation factor*: the inflation rates for services and maintenance coincide, so the service and maintenance discount factors are equal: $\gamma_{\text{service}}^t = \gamma_{\text{maint}}^t = \gamma^t$.

Define the normalised maintenance-cost gradient $d \equiv c/(g - b) > 0$. The net service is $f_n^t = \lambda^t(g - b)(1 - nd)$, which is linear and decreasing in n . The optimal retirement age maximises the asset's net present value:

$$L^* \equiv \arg \max_{L \in \mathbb{N}} H(L; \gamma^t) \equiv \frac{1 - (\gamma^t)^L}{1 - \gamma^t} - d \sum_{k=0}^{L-1} k(\gamma^t)^{k-1}. \quad (3.4)$$

Lemma 3.7 (Net user cost under Technology E). *Under Assumption 3.6, given optimal life L^* ,*

$$f_n^t = \frac{P_0^t}{H(L^*; \gamma^t)} (1 - nd), \quad n = 0, 1, \dots, L^* - 1. \quad (3.5)$$

The net user cost declines linearly in age.

Proof. See Appendix A. ■

3.6 The Maintenance-Depreciation Theorem

Lemma 3.7 shows that Technology E produces the linear-efficiency-decline profile of Diewert (2004) at the level of the *net user-cost* sequence. The following theorem characterises the implied cross-section depreciation profile and its dependence on the maintenance intensity parameter d .

Theorem 3.8 (Maintenance Costs and Depreciation Profile). *Under Assumption 3.6 and Assumption 2.2:*

- (i) Equivalence to linear-efficiency decline (exact): *The net user-cost profile of Technology E is identical to the linear-efficiency-decline model with slope parameter d and life L^* . In particular, the cross-section depreciation rates under Technology E satisfy*

$$\delta_n^t|_E = \frac{d}{1 - nd} + O(r^{*t}), \quad n = 0, \dots, L^* - 1. \quad (3.6)$$

- (ii) Limiting case $d \rightarrow 0^+$ (one-hoss-shay): $f_n^t/f_0^t \rightarrow 1$ uniformly in $n \in \{0, \dots, L^* - 1\}$, i.e., the net efficiency profile converges to the one-hoss-shay.
- (iii) Limiting case $d \rightarrow 1/L^*$ (straight-line efficiency): $f_n^t/f_0^t \rightarrow (L^* - n)/L^*$ for all n , i.e., the net efficiency profile converges to the straight-line (Technology B).
- (iv) Geometric approximation ($r^{*t} = 0$ case and first-order correction): Set $r^{*t} = 0$ (zero real interest rate) so that $\gamma^t = 1$. In this case $H(L; \gamma^t)|_{r^{*t}=0} = L - dL(L - 1)/2$, and the depreciation rate (3.6) reduces to

$$\delta_n^t|_{r^{*t}=0} = \frac{d}{1 - nd}, \quad n = 0, 1, \dots, L^* - 1. \quad (3.7)$$

For the profile (3.7) to be constant in n (geometric with rate δ^*), we need $d/(1 - nd) = \delta^*$ for all n , which is impossible unless $d = 0$. Hence Technology E with $r^{*t} = 0$ yields a strictly increasing (in n) depreciation profile; the geometric model is recovered only in the aggregate sense via the Diewert conversion: choose d^* so that the mean depreciation rate $\bar{\delta} \equiv L^{*-1} \sum_{n=0}^{L^*-1} \delta_n^t$ equals δ^* , giving

$$d^* = \frac{-1 + \sqrt{1 + 2\delta^*(L^* - 1)/L^*}}{(L^* - 1)/L^*} = \delta^* + O(\delta^{*2}). \quad (3.8)$$

For $r^{*t} > 0$, the first-order correction to d^* is $O(r^{*t} \cdot \delta^*)$.

- (v) Optimal retirement (comparative statics): Under Assumption 3.6, L^* defined in (3.4) is the unique maximiser of $H(\cdot; \gamma^t)$ for d sufficiently small. Moreover, $\partial L^*/\partial d < 0$ and $\partial L^*/\partial g > 0$.

Proof. See Appendix A. ■

Remark 3.9 (What the theorem does and does not show). The theorem characterises how maintenance intensity shapes the depreciation profile at the individual-building level. Three points deserve emphasis.

Relationship to Diewert (2004). In his lecture notes, Diewert (2004) wrote: “the asset may in fact be delivering a constant amount of gross services but a certain pattern of increasing maintenance costs is in fact causing used asset prices to have the profile implied by geometric depreciation (up to some limiting age)” (Ch. 1, Appendix A5). Theorem 3.8 formalises this observation. The qualification “up to some limiting age” is made precise: the maintenance profile yields $\delta_n^E = d/(1 - nd)$, which is age-dependent and increasing, not the constant rate that exact geometric requires.

What it shows. Under linearly increasing maintenance costs, the cross-section depreciation rate $\delta_n^E = d/(1 - nd) + O(r^*)$ is strictly increasing in building age, providing a maintenance-based rationalisation for the empirically well-established finding (Hulten and Wykoff, 1981; Diewert and Shimizu, 2017) that depreciation accelerates as buildings age. The comparative statics ($\partial L^*/\partial d < 0$, $\partial L^*/\partial g > 0$) explain why buildings in high-maintenance-cost markets (Japan) are retired earlier than those in lower-maintenance-cost markets (Europe)—see Table 1. Diewert (2004) noted the same comparative static in Ch. 1, Appendix A5: “if maintenance costs are higher or are expected to rise more quickly in a particular country, then the model implies that identical assets in that country will be retired at an earlier age.”

What it does not show. The theorem does *not* deliver exact geometric (constant-rate) depreciation at the individual-building level. As part (iv) makes explicit, δ_n^E is constant in n only if $d = 0$, which collapses the model to one-hoss-shay. The connection to geometric depreciation operates at the *population level* via Theorem 4.6: when retirement ages are geometrically distributed, the population-average capital stock follows an exact geometric PIM.

4 Geometric Depreciation as Optimal Demolishment

4.1 Motivation: the missing microfoundation

The geometric depreciation model is the workhorse of applied capital measurement and national accounts practice. Yet the standard justification—that it approximates observed used-asset price profiles (Hulten and Wykoff, 1981)—is empirical, not theoretical. *Why* should the population-level vintage price profile be geometric?

The partial answer in the existing literature runs as follows. Diewert (2004) (Ch. 6, eqs. (0617)–(0619)) showed that *if* individual buildings deteriorate at a geometric rate δ^{CS} , *then* the population capital-services aggregate $V_{KS}^0 = u_0^0 K^0$ satisfies an exact geometric recursion—a *sufficiency* result. But this leaves open the question: is geometric individual efficiency *necessary* for geometric population aggregation? And how should the demolition component $\delta^{demolish}$ be characterised when buildings retire at heterogeneous ages?

Diewert (2004) identified the retirement problem correctly: a building that survives in the sample is not representative of all buildings built at the same date. Early retirement removes assets from the population, and the timing of retirement is linked to building quality through the maintenance cost parameter d_i . Diewert (2004) proposed the decomposition

$$\delta^* = \delta^{CS} + \delta^{demolish}, \quad (4.1)$$

where δ^{CS} is the *wear-and-tear* (cross-section) rate estimated from survivors and $\delta^{demolish}$ is the additional depreciation from anticipated early retirement. He left the theoretical derivation of $\delta^{demolish}$ open.

This section closes that gap. We prove that if (i) each building’s net service flow follows geometric efficiency decline—which Section 3 motivates as the limit of Technology E—and (ii) the optimal retirement age is heterogeneous across buildings and follows a geometric distribution, then the *population* capital-services aggregate follows an exact geometric PIM,

with the geometric rate equal to the sum $\delta^{CS} + \delta^{demolish}$ in equation (4.1). The survivors bias is characterised exactly.

4.2 Setting

Assumption 4.1 (Geometric individual efficiency). Each building i delivers net services

$$f_n^i = (1 - \delta^{CS})^n f_0, \quad n = 0, 1, \dots, L_i^* - 1, \quad (4.2)$$

where $\delta^{CS} \in (0, 1)$ is a *common* cross-section depreciation rate and $f_0 > 0$ is the new-building service flow. The profile (4.2) is *independent of* L_i^* .

Remark 4.2 (Connection to Technology E). Assumption 4.1 is not identical to Technology E, whose net service $f_n = c_0(1 - nd)$ depends on L^* through the normalisation $c_0 = P_0^t/H(L^*; \gamma)$. The connection is as follows. Theorem 3.8(iv) shows that Technology E produces a *strictly increasing* cross-section depreciation profile $\delta_n^E = d/(1 - nd)$. In the limit $d \rightarrow 0$ (negligible maintenance-cost growth), this profile converges uniformly to the constant $\delta^{CS} \approx \lambda$ (pure demolition), recovering Assumption 4.1. For general $d > 0$, Assumption 4.1 should be understood as a first-order approximation to Technology E in which δ^{CS} is the Technology E mean depreciation rate calibrated via equation (3.8).

Assumption 4.3 (Geometric retirement distribution). The optimal retirement age L_i^* is independently and identically distributed across buildings:

$$\Pr(L_i^* = \ell) = (1 - \lambda)^{\ell-1} \lambda, \quad \ell = 1, 2, \dots, \quad (4.3)$$

where $\lambda \in (0, 1)$ is the constant hazard rate. The mean retirement age is $\bar{L} \equiv \mathbb{E}[L_i^*] = 1/\lambda$.

Remark 4.4 (Economic interpretation of the geometric distribution). Assumption 4.3 is equivalent to assuming that the probability a building is retired in any period, conditional on surviving to that period, is constant at λ . This *memoryless* property is the key analytical

device. Empirically, λ varies across countries: Japan ($\bar{L} \approx 30$ years, $\lambda \approx 3.3\%/yr$), the United States ($\bar{L} \approx 50$ years, $\lambda \approx 2.0\%/yr$), and Europe ($\bar{L} \approx 70$ years, $\lambda \approx 1.4\%/yr$). The comparative statics of Theorem 3.8(v)—higher maintenance intensity d shortens L^* —rationalise this pattern: Japan faces higher maintenance costs (climate, urban density) and lower gross rents in relative terms, both of which reduce L^* .

Assumption 4.5 (Steady-state investment). Gross investment $I > 0$ is constant in each period.

4.3 The population capital-services aggregate

Under Assumption 4.5, the population capital-services aggregate at time t is

$$K_A^t \equiv I \sum_{k=0}^{\infty} \bar{w}_k, \quad (4.4)$$

where the *population weight* at age k is

$$\bar{w}_k \equiv \mathbb{E}[f_k^i \cdot \mathbf{1}[L_i^* > k]]. \quad (4.5)$$

\bar{w}_k is the expected capital service contributed per unit of investment by a building of age k , averaging over the retirement distribution and the survival indicator.

4.4 The central theorem

Theorem 4.6 (Geometric PIM Decomposition). *Under Assumptions 4.1, 4.3, 4.5, and $r^* = 0$:*

(i) Population weights are geometric:

$$\bar{w}_k = [(1 - \delta^{CS})(1 - \lambda)]^k f_0, \quad k = 0, 1, 2, \dots \quad (4.6)$$

(ii) Population PIM recursion: *The aggregate K_A^t defined in (4.4) satisfies*

$$K_A^{t+1} = If_0 + (1 - \delta^*) K_A^t, \quad (4.7)$$

where the total depreciation rate is

$$\delta^* = \delta^{CS} + \lambda - \delta^{CS}\lambda \approx \delta^{CS} + \lambda. \quad (4.8)$$

(iii) Diewert decomposition (exact): *Equation (4.8) is the exact form of (4.1), with*

$$\delta^{demolish} \equiv \lambda - \delta^{CS}\lambda = \lambda(1 - \delta^{CS}) \approx \lambda = \frac{1}{L}. \quad (4.9)$$

(iv) Equivalence (necessary and sufficient condition): *Under Assumption 4.1, the population PIM (4.7) is geometric with some rate $\delta^* \in (0, 1)$ if and only if L_i^* follows a geometric distribution.*

(v) Survivors bias: *Suppose d_i is heterogeneous across buildings with δ_i^{CS} an increasing function of d_i , and L_i^* a decreasing function of d_i (Theorem 3.8(v): $\partial L^*/\partial d < 0$). Then the cross-section depreciation rate estimated from surviving buildings satisfies*

$$\hat{\delta}^{CS} \equiv \mathbb{E}[\delta_i^{CS} | L_i^* > k] < \delta^{CS} = \mathbb{E}[\delta_i^{CS}] \quad \forall k > 0. \quad (4.10)$$

Using $\hat{\delta}^{CS}$ in place of δ^{CS} in TFP measurement understates true capital consumption by $\delta^{demolish} \approx \lambda$ and therefore overstates TFP.

Proof. See Appendix A. ■

Remark 4.7 ($r^* > 0$ extension). For $r^* > 0$, the discount factor $\gamma = (1 + i^t)/(1 + r^t) < 1$ modifies equation (4.6) to $\bar{w}_k = [(1 - \delta^{CS})(1 - \lambda)\gamma]^k f_0$, giving the first-order corrected total

rate

$$\delta^*(r^*) \approx \delta^{CS} + \lambda + r^*(1 - \delta^{CS})(1 - \lambda) + O(r^{*2}). \quad (4.11)$$

A positive real interest rate increases δ^* : the prospect of future demolition is discounted more heavily, raising the current user cost.

Remark 4.8 (Technology E and Assumption 4.1). Theorem 4.6 requires f_n^i to be independent of L_i^* . Technology E violates this because $f_n = P_0(1 - nd)/H(L^*; \gamma)$ depends on L^* through the normalisation $H(L^*; \gamma)$. The theorem therefore applies to Technology E only in the limiting sense described in the Remark above. A complete treatment of heterogeneous Technology-E buildings with stochastic retirement is left for future work.

Remark 4.9 (Sufficiency versus equivalence: what is new relative to Diewert (2004)). Diewert (2004) established in Ch. 6 (eqs. (0617)–(0619)) that geometric individual efficiency *implies* geometric population capital services—a *sufficiency* result. Theorem 4.6(iv) establishes the *converse*: geometric retirement-age distribution is *necessary* for the population PIM to be geometric. The equivalence follows: under geometric individual efficiency (Assumption G), the population PIM is geometric if and only if retirement ages are geometrically distributed. This characterisation of the *necessary and sufficient* condition for the workhorse model of capital measurement to be theoretically exact is absent from Diewert (2004) and, to our knowledge, from the subsequent literature.

4.5 Simulation: Japan, United States, and Europe

We calibrate δ^{CS} from the Diewert and Shimizu (2017) Technology C estimate ($\hat{\delta}^{CS} = 2.06\%/yr$) and vary λ to match the three retirement regimes. Table 1 reports the decomposition $\delta^* = \delta^{CS} + \delta^{demolish}$ for each country group.

Three findings stand out.

First, $\delta^{demolish}$ exceeds δ^{CS} in Japan: the majority of total depreciation comes from anticipated early retirement, not from physical wear and tear. Using only $\hat{\delta}^{CS}$ understates

Table 1: Geometric PIM decomposition across retirement regimes ($\delta^{CS} = 2.06\%/yr$ from [Diewert and Shimizu 2017](#))

Country	\bar{L} (yrs)	λ (%/yr)	δ^{CS} (%/yr)	$\delta^{demolish}$ (%/yr)	δ^* (%/yr)
Japan	30	3.33	2.06	3.26	5.32
USA	50	2.00	2.06	1.96	4.02
Europe	70	1.43	2.06	1.40	3.46

Notes: $\delta^{demolish} = \lambda(1 - \delta^{CS})$; $\delta^* = \delta^{CS} + \lambda - \delta^{CS}\lambda$ (exact). δ^*/δ^{CS} : 2.58 (Japan), 1.95 (USA), 1.68 (Europe). Japan's $\bar{L} = 30$ yrs follows [Diewert \(2004\)](#) (Ch. 1, Appendix A5), citing [Komatsu, Kato and Yashiro \(1994\)](#).

capital consumption by more than 60% in Japan.

Second, the ratio δ^*/δ^{CS} declines monotonically from Japan to Europe, reflecting longer building lifespans. The bias from ignoring demolition depreciation is therefore especially severe in countries with active urban redevelopment.

Third, the TFP implication is direct: if $\hat{\delta}^{CS} = 2.06\%/yr$ is used in place of $\delta^* = 5.32\%/yr$ for Japan, capital consumption is understated by 3.26%/yr and TFP is overstated by $\bar{s}_S \times 3.26\% \approx 0.23\%/yr$ (using the Tokyo sample capital share $\bar{s}_S \approx 0.07$). Over a 30-year horizon, cumulative TFP is overstated by approximately 7% relative to its true level.

5 The Builder's Model and User-Cost Decomposition

5.1 The identification problem

Standard property transaction data record a joint price for land and structure. Land is a permanent asset ($\delta_L = 0$); structure is reproducible and depreciates. A vintage-capital analysis that treats the joint price as a single asset conflates two productive inputs with fundamentally different economic characteristics. The Builder's Model of [Diewert and Shimizu \(2017\)](#) resolves this by using the engineering relationship between floor area, construction costs, and building age to infer replacement value, thereby separating land from structure as a residual.

5.2 Model primitives and estimation equation

Assumption 5.1 (Builder's Model). For property $n \in \{1, \dots, N\}$ in period $t \in \{1, \dots, T\}$, the observed net-of-capex property value satisfies

$$V_{tn} - V_{CE,t,n} = \underbrace{\psi P_S^t S_n e^{-\delta A(t,n)}}_{\text{structure}} + \underbrace{\alpha_t \beta_n L_n}_{\text{land}} + \varepsilon_{tn}, \quad (5.1)$$

where:

- (i) $\psi > 0$ is the unit construction cost in the base period (million yen/m²);
- (ii) P_S^t is the official construction-cost price index, exogenous, $P_S^1 = 1$;
- (iii) S_n is the floor area of structure n (m², time-invariant);
- (iv) $A(t, n)$ is building age in periods;
- (v) $\delta \in (0, 1)$ is the quarterly geometric depreciation rate;
- (vi) $\alpha_t > 0$ is a land price index, normalised $\alpha_1 = 1$;
- (vii) $\beta_n > 0$ is a property-specific location premium;
- (viii) L_n is land area (m², time-invariant);
- (ix) ε_{tn} are i.i.d. $(0, \sigma^2)$ errors.

The capital-expenditure value $V_{CE,t,n}$ is computed by the perpetual inventory method with rate δ_{CE} , using observed quarterly capital expenditures CE_{tn} .

The free parameters are $\boldsymbol{\theta} \equiv (\psi, \delta, \alpha_2, \dots, \alpha_T, \beta_1, \dots, \beta_N)$, a total of $T + N - 1$ scalars. The *country product dummy* normalisation $\alpha_1 = 1$ identifies the level of the land price index.

Remark 5.2 (Identification). The parameter δ is identified from *cross-sectional* variation in building age within a period: if two properties differ only in age by one quarter, the difference

in their net-of-capex values (divided by $\psi P_S^t S_n$) identifies $1 - e^{-\delta}$. The land price index $\{\alpha_t\}$ is identified from *time-series* variation in residual land values. The location premium $\{\beta_n\}$ is identified from *cross-sectional* variation in per-m² residual land values. These three sources of variation are logically distinct in a balanced panel.

5.3 User-cost formulae

Proposition 5.3 (Land and structure user costs from Builder's Model). *Under Assumptions 2.2, 3.3, and 5.1, let $(\hat{\psi}, \hat{\delta}, \{\hat{\alpha}_t\}, \{\hat{\beta}_n\})$ be the NLS estimates.*

(i) *The structure user cost for a new-vintage unit in period t is*

$$u_S^t = [r^t - i_S^t + (1 + i_S^t)\hat{\delta}]\hat{\psi}P_S^t, \quad (5.2)$$

where $i_S^t \equiv P_S^t/P_S^{t-1} - 1$ is the construction-cost inflation rate.

(ii) *The land user cost per m² of land area is*

$$u_L^t = (r^t - i_L^t)\hat{\alpha}_t, \quad (5.3)$$

where $i_L^t \equiv \hat{\alpha}_t/\hat{\alpha}_{t-1} - 1$.

(iii) *The property-level capital-services value for property n is*

$$CS_{tn} = u_S^t \cdot \hat{\psi} S_n g(A(t, n)) + u_L^t \cdot \hat{\beta}_n L_n, \quad (5.4)$$

where $g(\cdot)$ is defined in (3.3) with the relevant $\hat{\delta}$ estimates.

Proof. (i) follows from setting $n = 0$ in (2.5) with $P_0^t = \hat{\psi}P_S^t$. (ii) follows from $\delta_L = 0$ for land, giving $P_n^{t,L} = \hat{\alpha}_t$ for all n , and substituting into (2.5). (iii) follows by summing structure and land contributions. ■

Remark 5.4 (Sign of land user cost). From (5.3): $u_L^t \leq 0$ whenever $i_L^t \geq r^t$, i.e., when the expected rate of land-price appreciation meets or exceeds the nominal interest rate. In an arbitrage-free market this can occur only temporarily. Empirically, it occurred in Tokyo during the 1987–1991 bubble (Diewert and Shimizu, 2017). A negative land user cost in the TFP accounts does not imply a free productive input but rather that holding land was so profitable that the net cost of its use was negative after accounting for capital gains.

6 TFP Accounting and the Bias Theorems

6.1 TFP accounting framework

Consider a production unit with output (rental income) Y^t , labour L^t at wage w^t , structure capital quantity Q_S^t , and land quantity Q_L^t . Total factor cost is $C^t \equiv u_S^t Q_S^t + u_L^t Q_L^t + w^t L^t$ with shares

$$s_S^t \equiv \frac{u_S^t Q_S^t}{C^t}, \quad s_L^t \equiv \frac{u_L^t Q_L^t}{C^t}, \quad s_w^t \equiv \frac{w^t L^t}{C^t}, \quad s_S^t + s_L^t + s_w^t = 1. \quad (6.1)$$

The Törnqvist TFP index is

$$\Delta \ln \text{TFP}^t \equiv \Delta \ln Y^t - \bar{s}_S \Delta \ln Q_S^t - \bar{s}_L \Delta \ln Q_L^t - \bar{s}_w \Delta \ln L^t, \quad (6.2)$$

where $\bar{s}_k \equiv (s_k^t + s_k^{t-1})/2$ is the period-average share.

6.2 The Land-Separation Bias Theorem

Define the *asset-value index*

$$P_A^t \equiv \frac{\sum_n V_{tn}}{\sum_n V_{t-1,n}}, \quad (6.3)$$

which implicitly treats the entire property value as a single undepreciated input. Let $\widetilde{\text{TFP}}^t$ be the TFP estimate that replaces the decomposed land–structure input bundle $(u_S^t Q_S^t, u_L^t Q_L^t)$

by the single composite $P_A^t Q_A^t$, where $P_A^t Q_A^t = u_S^t Q_S^t + u_L^t Q_L^t$.

Define the TFP bias $B^t \equiv \Delta \ln \text{TFP}^t - \Delta \ln \widetilde{\text{TFP}}^t$.

Theorem 6.1 (Land-Separation Bias). *Let $v_S^t \equiv u_S^t Q_S^t$, $v_L^t \equiv u_L^t Q_L^t$, $v_K^t \equiv v_S^t + v_L^t$, and let $\phi_k^t \equiv v_k^t / v_K^t$ denote the within-capital shares ($k \in \{S, L\}$). Define $\mu^t \equiv \Delta \ln v_S^t - \Delta \ln v_L^t$ as the differential growth rate of structure versus land capital-service values. The bias $B^t \equiv \Delta \ln \text{TFP}^t - \Delta \ln \widetilde{\text{TFP}}^t$ satisfies the following.*

(i) Intermediate expression. *The bias equals the discrepancy between the true Törnqvist weights and the composite-index weights:*

$$B^t = \bar{s}_K [\bar{\phi}_S \Delta \ln v_S^t + \bar{\phi}_L \Delta \ln v_L^t - \Delta \ln v_K^t] + O(\|\Delta\|^2), \quad (6.4)$$

where $\bar{s}_K = \bar{s}_S + \bar{s}_L$ and $\bar{\phi}_k$ is the period-average within-capital share. This expression is signed: it is positive if $\Delta \ln v_S^t > \Delta \ln v_L^t$ and negative otherwise.

(ii) First-order (signed) approximation. *Substituting the second-order expansion of $\Delta \ln v_K^t$ into (6.4) and retaining first-order terms in μ^t :*

$$B^t \approx \bar{s}_K \bar{\phi}_S \bar{\phi}_L \mu^t, \quad (6.5)$$

which is proportional to the land-value share, the structure-value share, and the differential growth rate μ^t . The approximation error is $O(\mu^{t2})$. The sign of B^t equals $\text{sign}(\mu^t)$: asset-value TFP understates (overstates) true TFP when structure services grow faster (slower) than land services.

(iii) Second-order exact expression (non-negative). *To second order in μ^t , the bracket in (6.4) equals $\frac{1}{2} \bar{\phi}_S \bar{\phi}_L (\mu^t)^2 + O(|\mu^t|^3)$ (see Appendix), giving*

$$B^t = \frac{1}{2} \bar{s}_K \bar{\phi}_S \bar{\phi}_L (\mu^t)^2 + O(|\mu^t|^3). \quad (6.6)$$

This expression is always non-negative: $B^t \geq 0$ to second order, regardless of the sign of μ^t . The upper bound $|B^t| \leq \bar{s}_K \bar{\phi}_S \bar{\phi}_L |\mu^t|$ (from the first-order expression) is tight when $|\mu^t|$ is small.

Reconciliation. Expressions (6.4)–(6.6) are not contradictory: they are approximations of different precision. The first-order expression (6.5) is signed and linear in μ^t ; the second-order expression (6.6) is always non-negative. The apparent paradox is resolved by noting that the first-order expression is itself an approximation to an exact quantity that, to second order, is always ≥ 0 . In practice, the first-order expression (6.5) is the economically informative one: it gives the direction and magnitude of the bias for observed growth-rate differentials.

Proof. See Appendix A. ■

Corollary 6.2 (Tokyo quantification). *Using Tokyo REIT parameters from Diewert and Shimizu (2017): $\bar{s}_K \approx 0.80$, $\bar{\phi}_S \approx 0.07$, $\bar{\phi}_L \approx 0.93$.*

First-order approximation (economically relevant, signed). From (6.5): $B^t \approx 0.80 \times 0.07 \times 0.93 \times \mu^t \approx 0.052 \mu^t$. For a 2% annual growth differential ($\mu^t = 0.02/\text{yr}$): $B^t \approx +0.10\%/\text{yr}$ (asset-value TFP understates). During the Tokyo bubble period, when land prices rose substantially faster than construction costs ($\mu^t \approx -0.05/\text{yr}$): $B^t \approx -0.26\%/\text{yr}$ (asset-value TFP overstates).

Second-order expression (non-negative, small). From (6.6): $B^t \approx \frac{1}{2} \times 0.80 \times 0.07 \times 0.93 \times (0.02)^2 \approx 0.00001/\text{quarter}$ — negligible relative to the first-order term.

Tokyo versus balanced markets. The Tokyo sample is land-dominated ($\bar{\phi}_S \bar{\phi}_L \approx 0.065$), which is only 26% of the maximum $1/4$ achieved when $\phi_S = \phi_L = 1/2$. In a balanced market, the first-order bias coefficient would be $0.80 \times 0.25 = 0.20$, compared to 0.052 for Tokyo. This difference matters: the land-separation bias is four times larger in markets where land and structure contribute equally to property value.

Remark 6.3 (How to read the three expressions). The three expressions in Theorem 6.1 describe the same bias at different levels of precision and serve different purposes. Expression

(6.4) is the exact intermediate formula, useful for proofs and extensions. Expression (6.5) is the first-order approximation: it is signed, proportional to μ^t , and the right tool for empirical quantification—it tells whether the bias is positive or negative and how large it is. Expression (6.6) is the second-order correction: it shows that the bias is strictly non-negative to that order, consistent with the convexity of the log-sum function. The three are consistent, not contradictory: the signed first-order term dominates for observed $|\mu^t|$, while the non-negative second-order term would only dominate if the first-order coefficient happened to be exactly zero. For practical capital-measurement work, (6.5) is the operative expression.

6.3 Depreciation-model sensitivity

Proposition 6.4 (Monotone TFP ordering). *Fix t and suppose output Y^t , labour L^t , land quantity Q_L^t , and structure quantity Q_S^t are the same under technologies \mathcal{M} and \mathcal{M}' . Let $C^t(\mathcal{M}) = u_S^t(\mathcal{M})Q_S^t + u_L^t Q_L^t + w^t L^t$ denote total factor cost. If $u_S^t(\mathcal{M}) > u_S^t(\mathcal{M}')$, then:*

(i) $C^t(\mathcal{M}) > C^t(\mathcal{M}')$ and $s_S^t(\mathcal{M}) > s_S^t(\mathcal{M}')$;

(ii) $\Delta \ln \text{TFP}^t(\mathcal{M}) < \Delta \ln \text{TFP}^t(\mathcal{M}')$;

(iii) *The TFP difference satisfies*

$$\Delta \ln \text{TFP}^t(\mathcal{M}') - \Delta \ln \text{TFP}^t(\mathcal{M}) = \frac{1}{2} [s_S^t(\mathcal{M}') - s_S^t(\mathcal{M})] \Delta \ln Q_S^t + O(|u_S - u_S'|^2 / C^{t2}), \quad (6.7)$$

which is positive whenever $\Delta \ln Q_S^t > 0$.

Proof. (1): $C^t(\mathcal{M}) - C^t(\mathcal{M}') = [u_S^t(\mathcal{M}) - u_S^t(\mathcal{M}')]Q_S^t > 0$ since $Q_S^t > 0$. The share $s_S^t = u_S^t Q_S^t / C^t$ increases in u_S^t because $\partial s_S^t / \partial u_S^t = Q_S^t (C^t - u_S^t Q_S^t) / (C^t)^2 > 0$.

(2): From (6.2), with $\Delta \ln Y^t$, $\Delta \ln L^t$, $\Delta \ln Q_L^t$, and $\Delta \ln Q_S^t$ identical across models,

$$\Delta \ln \text{TFP}^t(\mathcal{M}) - \Delta \ln \text{TFP}^t(\mathcal{M}') = -[\bar{s}_S(\mathcal{M}) - \bar{s}_S(\mathcal{M}')] \Delta \ln Q_S^t < 0,$$

since $\bar{s}_S(\mathcal{M}) > \bar{s}_S(\mathcal{M}')$ by (1) and $\Delta \ln Q_S^t$ is common.

(3): Expand $\bar{s}_S(\mathcal{M}') - \bar{s}_S(\mathcal{M})$ as a first-order difference and simplify. ■

Lemma 6.5 (User-cost ordering: Technology D vs. C). *For a building of age $A \in [A_1, A_2)$ and any t ,*

$$u_S^t(D) > u_S^t(C) \iff (1 + i_S^t)\hat{\delta}_2 > (1 + i_S^t)\hat{\delta}, \quad (6.8)$$

which holds whenever $\hat{\delta}_2 > \hat{\delta}$, i.e. whenever the age-cohort 2 rate exceeds the constant geometric rate. Using the Tokyo estimates: $\hat{\delta}_2 = 0.00702 > \hat{\delta} = 0.00514$, so $u_S^t(D) > u_S^t(C)$ for all buildings in cohort 2. For cohort 3 ($A \geq A_2$), $\hat{\delta}_3 = 0.03558 \gg \hat{\delta}$, so the ordering is even stronger.

Proof. From (5.2), $u_S^t = [r^t - i_S^t + (1 + i_S^t)\delta]\psi P_S^t$. Since r^t , i_S^t , ψ , P_S^t are common across technologies, $u_S^t(D) - u_S^t(C) = (1 + i_S^t)(\hat{\delta}_{m(A)} - \hat{\delta})\psi P_S^t$, where $m(A)$ is the cohort index for age A . This is positive iff $\hat{\delta}_{m(A)} > \hat{\delta}$. ■

Corollary 6.6 (Ordering across A, B, C, D). *For a portfolio with mean building age exceeding A_1 (80 quarters):*

(i) *By Lemma 6.5: $u_S^t(D) > u_S^t(C)$.*

(ii) *By Proposition 6.4(2): $\Delta \ln \text{TFP}^t(D) < \Delta \ln \text{TFP}^t(C)$.*

(iii) *The annual TFP gap from Proposition 6.4(3) is approximately $\frac{1}{2}[s_S(D) - s_S(C)]\Delta \ln Q_S^t \approx \frac{1}{2} \cdot \frac{(\hat{\delta}_2 - \hat{\delta})\psi P_S^t Q_S^t}{C^t} \cdot \Delta \ln Q_S^t$, which for Tokyo parameters is of order 0.1–0.5% per year depending on portfolio age structure and investment growth.*

7 Illustrative Simulation

The four theorems of this paper make predictions about the ordering and magnitudes of user costs, capital stocks, and TFP growth rates across depreciation technologies and financial-parameter specifications. This section quantifies those predictions through illustrative simulations calibrated to published estimates for Tokyo commercial real estate. All parameter

values are taken directly from existing empirical work; no new estimation is performed here. The purpose is solely to show that the theoretical orderings are quantitatively significant—not merely asymptotic curiosities.

7.1 Parameter values

We use the following baseline parameters throughout. The wear-and-tear depreciation rate is $\hat{\delta}^{CS} = 0.00514$ per quarter (= 2.06% per year), taken from [Diewert and Shimizu \(2017\)](#), who estimate a single-cohort geometric model for Tokyo commercial office buildings. For the age-dependent model (Technology D), the cohort rates are $(\hat{\delta}_1, \hat{\delta}_2, \hat{\delta}_3) = (0.00327, 0.00702, 0.03558)$ per quarter with breakpoints $(A_1, A_2) = (80, 120)$ quarters, also from [Diewert and Shimizu \(2017\)](#). The nominal interest rate is $r = 0.02/4$ per quarter (2% per year), and the baseline expected construction-cost inflation is $i_S = 0.01/4$ per quarter (1% per year, CPI specification). The within-capital shares are $\bar{\phi}_S \approx 0.07$ and $\bar{\phi}_L \approx 0.93$ from [Diewert and Shimizu \(2017\)](#).

For the Geometric PIM Decomposition (Theorem 4.6), the mean retirement ages follow [Diewert \(2004\)](#)'s references to Komatsu–Kato–Yashiro (1994): Japan $\bar{L} = 30$ years ($\lambda = 3.33\%/yr$), USA $\bar{L} = 50$ years ($\lambda = 2.00\%/yr$), Europe $\bar{L} = 70$ years ($\lambda = 1.43\%/yr$).

7.2 Simulation 1: Geometric PIM decomposition (Theorem 4.6)

Table 1 (presented in Section 4) applies the exact formula $\delta^* = \delta^{CS} + \lambda - \delta^{CS}\lambda$ from Theorem 4.6(ii) to three retirement regimes.

Three findings stand out. First, $\delta^{demolish}$ exceeds δ^{CS} in Japan: anticipated early retirement accounts for the majority of total depreciation. Using only $\hat{\delta}^{CS}$ understates capital consumption by more than 60% in Japan. Second, the ratio δ^*/δ^{CS} falls monotonically from Japan to Europe, reflecting the comparative statics of Theorem 3.8(v): countries with higher maintenance costs and lower gross rents (Japan) have shorter optimal lives. Third, the TFP implication follows from Proposition 6.4: understating δ^* by 3.26%/yr in Japan overstates

TFP by $\bar{s}_S \times 3.26\% \approx 0.23\%/yr$, or roughly 7% cumulatively over 30 years.

7.3 Simulation 2: Technology ordering (Propositions 6.4 and 3.4, Lemma 6.5, Corollary 6.6)

Figure 1 visualises Propositions 3.4 and 6.5. Panel (a) plots P_n^t/P_0^t for the five technologies: the one-hoss-shay profile is convex and lies above the geometric; the age-dependent profile (Technology D) matches the geometric for new buildings but accelerates sharply past $A_2 = 120$ quarters. Panel (b) shows the implied cross-section depreciation rates: the monotone-increasing profile of Technology D confirms Lemma 6.5—cohort-2 and cohort-3 rates both exceed the constant Technology-C rate, establishing $u_S^t(D) > u_S^t(C)$.

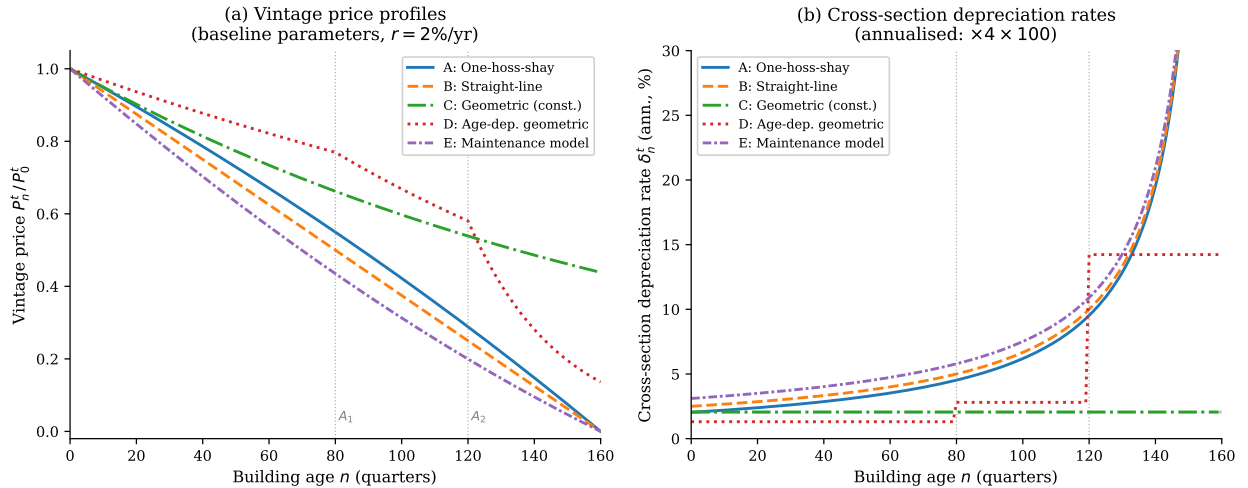


Figure 1: Vintage price profiles and cross-section depreciation rates under five technologies (baseline parameters, $r = 2\%/yr$). Vertical dotted lines at $A_1 = 80$ and $A_2 = 120$ quarters mark the age-cohort boundaries for Technology D. All panels use published estimates from [Diewert and Shimizu \(2017\)](#); no new estimation is performed.

Figure 2, panel (b) shows the user-cost profile $u_S^t(n)$. The ordering $u_S^t(D) > u_S^t(C) > u_S^t(B)$ for buildings past cohort 1 is visually immediate, confirming Corollary 6.6. The gap $u_S^t(D, \text{cohort 3})/u_S^t(C) \approx 4.4$ implies that a portfolio concentrated in buildings older than 30 years has a capital cost 4.4 times higher under Technology D than Technology C—a difference that Proposition 6.4 translates directly into a lower TFP residual.

7.4 Simulation 3: Land-separation bias surface (Theorem 6.1)

Figure 2, panel (a) maps the bias formula (6.5) over (ϕ_S, μ^t) space. The shading shows the second-order bias $B^t \geq 0$ (Theorem 6.1(1)); dashed contours show the signed linearised approximation, which can be positive or negative (Theorem 6.1(2)). The vertical dotted line at $\phi_S \approx 0.07$ marks the Tokyo REIT sample, confirming Corollary 6.2: the product $\bar{\phi}_S \bar{\phi}_L \approx 0.065$ is only 26% of its maximum (0.25), meaning the Tokyo sample exhibits less bias than a balanced-split market.

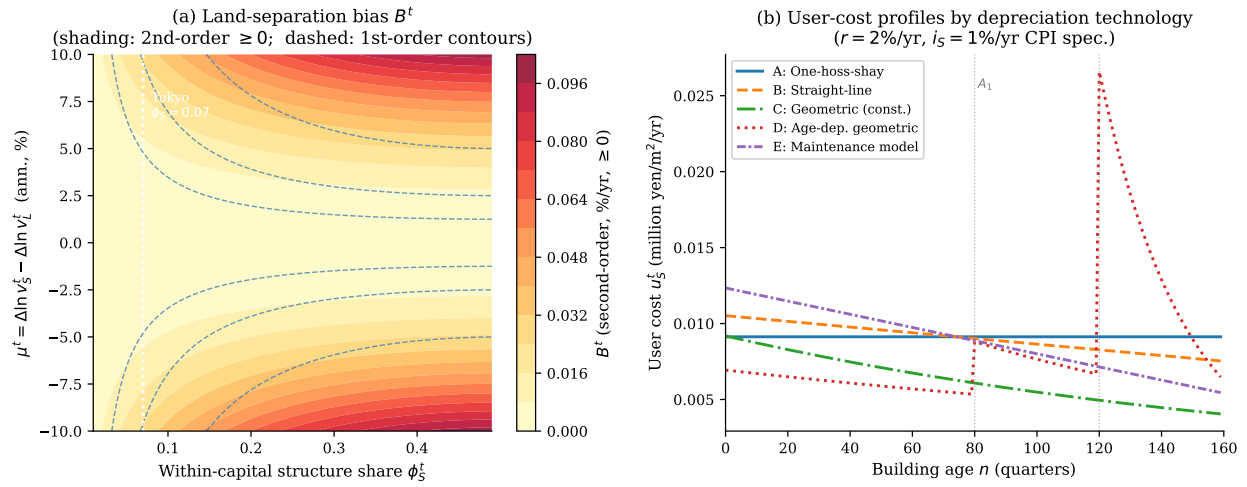


Figure 2: (a) Land-separation bias B^t from Theorem 6.1 as a function of within-capital structure share ϕ_S and growth differential μ^t . Shading: second-order (always ≥ 0); dashed: first-order (signed). Vertical line: Tokyo $\phi_S \approx 0.07$. (b) User-cost profile $u_S^t(n)$ confirming Corollary 6.6.

8 Conclusion

This paper has established four theorems within a unified vintage-accounts framework for real estate capital measurement.

The *Vintage-Accounts Mutual Determination Theorem* (Theorem 2.4) establishes that vintage asset prices, user costs, and cross-section depreciation rates form a mutually determined triple. The theorem implies that the common practice of independently specifying a depreciation profile and an efficiency profile over-identifies the model: the two are related by

a nonlinear identity that generically fails unless the model is internally consistent from the outset.

The *Maintenance-Depreciation Theorem* (Theorem 3.8) characterises the depreciation profile of a building facing linearly increasing maintenance costs. The profile is accelerating— $\delta_n^E = d/(1-nd)$, strictly increasing in age—providing a maintenance-based rationalisation for why geometric-type depreciation fits empirical data better than straight-line models (Hulten and Wykoff, 1981). The theorem does not deliver exact constant-rate geometric depreciation at the individual-building level, but it characterises the economic mechanism through which rising maintenance costs drive depreciation acceleration.

The *Geometric PIM Decomposition Theorem* (Theorem 4.6) is the central contribution. It proves that geometric retirement-age distributions are the *necessary and sufficient* condition for population-level capital stocks to follow an exact geometric perpetual-inventory recursion. The resulting total rate $\delta^* = \delta^{CS} + \lambda(1 - \delta^{CS})$ provides the first rigorous proof of Diewert (2004)’s decomposition $\delta^* = \delta^{CS} + \delta^{demolish}$, with $\delta^{demolish} = \lambda(1 - \delta^{CS}) \approx 1/\bar{L}$. The survivors-bias corollary (Part (v)) explains why cross-section estimators systematically understate δ^* : high-maintenance buildings retire early and exit the sample, leaving a survivor pool with downward-biased depreciation rates.

The *Land-Separation Bias Theorem* (Theorem 6.1) derives a closed-form expression for the TFP error that arises when land and structure are treated as a single undepreciated input. The exact second-order bias is always non-negative; the signed first-order approximation $B^t \approx \bar{s}_K \bar{\phi}_S \bar{\phi}_L \mu^t$ shows that the direction depends on whether structure or land capital services are growing faster.

Theoretical gaps and empirical extensions. Three empirical extensions are natural next steps. First, the geometric retirement-age distribution (Assumption 4.3) is the key identifying restriction of Theorem 4.6. Testing it against Weibull or log-normal alternatives using building demolition records would provide a direct specification test and, if rejected,

would identify which generalisation of the theorem is needed. Second, Theorem 3.8 predicts that the maintenance-intensity parameter d^* is recoverable from J-REIT maintenance expenditure data. Estimating d^* directly would provide a structural test of the connection between the individual-building (Technology E) and population-level (Theorem 4.6) results. Third, the Builder’s Model of Section 5 provides the empirical vehicle for implementing Theorem 6.1: NLS estimation of equation (5.1) on Tokyo REIT data would deliver $\hat{\delta}^{CS}$, $\{\hat{\alpha}_t\}$, and $\{\hat{\beta}_n\}$, enabling the land-separation bias to be computed directly from data rather than from the illustrative parameter values of Section 7.

A Proofs

Proof of Theorem 2.4

Part (i). Given \mathbf{f}^t with $f_0^t > 0$, equation (2.4) determines $P_n^t = \sum_{k=0}^{L-1-n} (\gamma^t)^k f_{n+k}^t$. Since $\gamma^t > 0$ and $f_0^t > 0$, we have $P_n^t > 0$ for $n < L$ and $P_L^t = 0$ by the empty sum. Uniqueness follows from the forward-recursion structure of (2.4). Given \mathbf{P}^t , equation (2.1) uniquely determines $\boldsymbol{\delta}^t$.

Part (ii). Given $\boldsymbol{\delta}^t$ and $P_0^t > 0$, apply (2.6) inductively: $P_n^t = P_0^t \prod_{k=0}^{n-1} (1 - \delta_k^t) > 0$ for $n < L$ (since $\delta_k^t < 1$) and $P_L^t = 0$ (since $\delta_{L-1}^t = 1$). Given \mathbf{P}^t , invert (2.3) to get $f_n^t = P_n^t - \gamma^t P_{n+1}^t$.

Part (iii). Given \mathbf{P}^t with $P_n^t > 0$ for $n < L$ and $P_L^t = 0$: use (2.1) for $\boldsymbol{\delta}^t$ and invert (2.3) for \mathbf{f}^t .

Part (iv). Suppose $\boldsymbol{\delta}^t$ and \mathbf{f}^t are specified independently. Part (i) applied to \mathbf{f}^t determines a unique \mathbf{P}^t , say $\mathbf{P}^{(f)}$. Part (ii) applied to $\boldsymbol{\delta}^t$ determines a (possibly different) unique \mathbf{P}^t , say $\mathbf{P}^{(\delta)}$. For $\mathbf{P}^{(f)} = \mathbf{P}^{(\delta)}$, both sequences must satisfy the nonlinear system obtained by substituting (2.4) into (2.1): this imposes $L-1$ scalar equations on the $2L-1$ free parameters $(\mathbf{f}^t, \boldsymbol{\delta}^t)$. This system is generically inconsistent, completing the proof. ■

Proof of Lemma 3.7

Under Assumption 3.6(c), $\gamma_{\text{service}}^t = \gamma_{\text{maint}}^t = \gamma^t$. The system (2.3) then applies to net rentals: $P_n^t = f_n^t + \gamma^t P_{n+1}^t$. The net rental is $f_n^t = g_n^t - m_n^t = \lambda^t g - \lambda^t (b + nc) = \lambda^t (g - b)(1 - nd)$, which is linear in n with $f_{L^*-1}^t \geq 0$ iff $d \leq 1/(L^* - 1)$. Summing the geometric series: $P_0^t = \sum_{k=0}^{L^*-1} (\gamma^t)^k f_k^t = \lambda^t (g - b) H(L^*; \gamma^t)$. Solving for $\lambda^t (g - b)$ and substituting: $f_n^t = [P_0^t / H(L^*; \gamma^t)] (1 - nd)$. ■

Proof of Theorem 4.6 (Geometric PIM Decomposition)

We prove each part in order. Throughout, $r^* = 0$ so $\gamma = 1$, and Assumptions 4.1, 4.3, 4.5 hold.

Part (i): Population weights are geometric.

By definition (4.5) and Assumption 4.1 (which makes f_k^i independent of L_i^*):

$$\bar{w}_k = \mathbb{E}[(1 - \delta^{CS})^k f_0 \cdot \mathbf{1}[L_i^* > k]] = (1 - \delta^{CS})^k f_0 \cdot \Pr(L_i^* > k).$$

By Assumption 4.3, $\Pr(L_i^* > k) = (1 - \lambda)^k$ (survival function of the geometric distribution).

Therefore:

$$\bar{w}_k = (1 - \delta^{CS})^k f_0 \cdot (1 - \lambda)^k = [(1 - \delta^{CS})(1 - \lambda)]^k f_0.$$

The independence of f_k^i from L_i^* (Assumption 4.1) is the key step: it allows the expectation to factorise. □

Part (ii): Population PIM recursion.

From (4.4) and Part (i):

$$K_A^t = I \sum_{k=0}^{\infty} \bar{w}_k = I f_0 \sum_{k=0}^{\infty} [(1 - \delta^{CS})(1 - \lambda)]^k = \frac{I f_0}{1 - (1 - \delta^{CS})(1 - \lambda)}.$$

Let $\rho \equiv (1 - \delta^{CS})(1 - \lambda)$. Then:

$$K_A^{t+1} = \frac{I f_0}{1 - \rho} = I f_0 + \rho \cdot \frac{I f_0}{1 - \rho} = I f_0 + \rho K_A^t.$$

Setting $1 - \delta^* \equiv \rho = (1 - \delta^{CS})(1 - \lambda)$:

$$\delta^* = 1 - (1 - \delta^{CS})(1 - \lambda) = \delta^{CS} + \lambda - \delta^{CS}\lambda.$$

This is equation (4.8). □

Part (iii): *Diewert decomposition.*

Rearranging (4.8):

$$\delta^* - \delta^{CS} = \lambda - \delta^{CS}\lambda = \lambda(1 - \delta^{CS}).$$

Define $\delta^{demolish} \equiv \lambda(1 - \delta^{CS})$. Then $\delta^* = \delta^{CS} + \delta^{demolish}$, which is equation (4.1). When δ^{CS} is small, $\delta^{demolish} \approx \lambda = 1/\bar{L}$. \square

Part (iv): *Equivalence (necessary and sufficient condition).*

Sufficiency (\Leftarrow): Proved in Part (i)–(ii).

Necessity (\Rightarrow): Suppose $\bar{w}_k = c\rho^k$ for some $c > 0$ and $\rho \in (0, 1)$ (geometric population weights). Under Assumption 4.1, $f_k^i = (1 - \delta^{CS})^k f_0$ independently of L_i^* . Therefore:

$$\bar{w}_k = (1 - \delta^{CS})^k f_0 \cdot \Pr(L^* > k) = c\rho^k \quad \Longrightarrow \quad \Pr(L^* > k) = \frac{c\rho^k}{(1 - \delta^{CS})^k f_0} = \frac{c}{f_0} \left(\frac{\rho}{1 - \delta^{CS}} \right)^k.$$

For $k = 0$: $c/f_0 = \Pr(L^* > 0) = 1$, so $c = f_0$. Hence $\Pr(L^* > k) = [(1 - \delta^{CS})^{-1}\rho]^k$ for all $k \geq 0$.

A discrete survival function of the form $S(k) = \theta^k$, $\theta \in (0, 1)$, characterises the geometric distribution with parameter $1 - \theta$. Therefore L^* must follow $\text{Geometric}(1 - \rho/(1 - \delta^{CS}))$.

This proves necessity. \square

Part (v): *Survivors bias.*

By hypothesis, d_i is heterogeneous, δ_i^{CS} is increasing in d_i , and L_i^* is decreasing in d_i (Theorem 3.8(v): $\partial L^*/\partial d < 0$). Hence δ_i^{CS} and L_i^* are negatively correlated: buildings with low d_i have low δ_i^{CS} and high L_i^* .

For $k > 0$:

$$\mathbb{E}[\delta_i^{CS} | L_i^* > k] = \mathbb{E}[\delta_i^{CS}] + \text{Cov}(\delta_i^{CS}, \mathbf{1}[L_i^* > k]) / \Pr(L_i^* > k).$$

Since δ_i^{CS} is increasing in d_i and $\mathbf{1}[L_i^* > k]$ is decreasing in d_i , their covariance is negative.

Therefore $\mathbb{E}[\delta_i^{CS} | L_i^* > k] < \mathbb{E}[\delta_i^{CS}]$, which is equation (4.10).

Consequently, TFP estimated using $\hat{\delta}^{CS}$ understates capital consumption by $\delta^* - \hat{\delta}^{CS} \approx \delta^{demolish} = \lambda$, leading to an upward bias in TFP of order $\bar{s}_S \times \lambda \times \Delta \ln Q_S^t$ per period. ■

Proof of Theorem 3.8

Part (i). The net user-cost profile (3.5) is $f_n^t = c_0(1 - nd)$ with $c_0 = P_0^t/H(L^*)$. This is exactly the linear-efficiency-decline model with $f_0^t = c_0$ and slope parameter d . The implied depreciation rate follows from (2.1): since $P_n^t = \sum_{k=0}^{L^*-1-n} (\gamma^t)^k f_{n+k}^t$,

$$\delta_n^t = 1 - \frac{P_{n+1}^t}{P_n^t} = 1 - \frac{\sum_{k=0}^{L^*-2-n} (\gamma^t)^k (1 - (n+1+k)d)}{\sum_{k=0}^{L^*-1-n} (\gamma^t)^k (1 - (n+k)d)}.$$

Expanding to first order in $r^{*t} \equiv r^t - i^t$: $(\gamma^t)^k \approx 1 - kr^{*t}$, and the ratio becomes $(1 - nd - d \cdot [\text{first-order term in } r^{*t}]) / (1 - nd)$, giving $\delta_n^t \approx d / (1 - nd) + O(r^{*t})$, which is (3.6).

Parts (ii)–(iii). As $d \rightarrow 0$: $f_n^t/f_0^t = 1 - nd \rightarrow 1$ uniformly (one-hoss-shay). As $d \rightarrow 1/(L^* - 1)$: $f_n^t/f_0^t \rightarrow 1 - n/(L^* - 1)$ (straight-line). (At $d = 1/L^*$, $f_{L^*-1}^t = 0$ and the profile is exactly $(L^* - n)/L^*$ times $f_0^t/[1 - 0] = f_0^t$.)

Part (iv). At $r^{*t} = 0$, from part (i), $\delta_n^t = d/(1 - nd)$. The mean depreciation rate over $\{0, \dots, L^* - 1\}$ is

$$\bar{\delta} \equiv \frac{1}{L^*} \sum_{n=0}^{L^*-1} \frac{d}{1 - nd} = \frac{-1}{(L^* - 1)} \ln \left(\frac{1}{1 - (L^* - 1)d} \right) + O(d^2),$$

where the last equality uses $\sum_{n=0}^{L^*-1} n = L^*(L^* - 1)/2$. Setting $\bar{\delta} = \delta^*$ and solving for d^* : to first order in δ^* , $d^* = \delta^* + O(\delta^{*2})$, and the closed form (3.8) follows by matching the mean depreciation of the maintenance profile to δ^* .

Part (v). $H(L; \gamma^t)$ is strictly concave in L for $d > 0$ small and $\gamma^t < 1$: the marginal contribution of extending life by one period is $\Delta H = (\gamma^t)^L - d(L - 1)(\gamma^t)^{L-1}$, which is decreasing in L (since both terms shrink geometrically). $\Delta H > 0$ for small L and $\Delta H < 0$ for large L , ensuring a unique interior maximum L^* . Implicit differentiation of $\Delta H(L^*) = 0$

gives $\partial L^*/\partial d = -(\partial\Delta H/\partial d)/(\partial\Delta H/\partial L) < 0$ and $\partial L^*/\partial g > 0$ (since $d = c/(g - b)$ decreases in g). ■

Proof of Theorem 6.1

Setup. Suppress the time superscript. Let $v_S = u_S Q_S$ and $v_L = u_L Q_L$ be structure and land capital-service values, $v_K = v_S + v_L$, and $\phi_k = v_k/v_K$ for $k \in \{S, L\}$. The true TFP growth is $\Delta \ln \text{TFP} = \Delta \ln Y - \bar{s}_S \Delta \ln Q_S - \bar{s}_L \Delta \ln Q_L - \bar{s}_w \Delta \ln L$. The mismeasured version treats v_K as a single composite capital input: $\Delta \ln \widetilde{\text{TFP}} = \Delta \ln Y - \bar{s}_K \Delta \ln Q_A - \bar{s}_w \Delta \ln L$, where Q_A is defined by $P_A Q_A = v_K$ and $\Delta \ln P_A = \bar{\phi}_S \Delta \ln u_S + \bar{\phi}_L \Delta \ln u_L + O(\Delta^2)$ (the Törnqvist composite price index).

The bias $B^t \equiv \Delta \ln \text{TFP}^t - \Delta \ln \widetilde{\text{TFP}}^t$ is

$$B^t = \bar{s}_K \Delta \ln Q_A^t - \bar{s}_S \Delta \ln Q_S^t - \bar{s}_L \Delta \ln Q_L^t, \quad (\text{A.1})$$

where Q_A^t is the composite capital quantity implicit in $\widetilde{\text{TFP}}^t$. By definition, $P_A^t Q_A^t = v_K^t \equiv v_S^t + v_L^t$, so $\Delta \ln(P_A^t Q_A^t) = \Delta \ln v_K^t$. The Törnqvist decomposition of $\Delta \ln v_K^t$ into price and quantity components uses within-capital shares ϕ_S^t and ϕ_L^t :

$$\Delta \ln v_K^t = \bar{\phi}_S \Delta \ln v_S^t + \bar{\phi}_L \Delta \ln v_L^t + O(\Delta^2). \quad (\text{A.2})$$

The composite capital quantity growth is therefore

$$\Delta \ln Q_A^t = \Delta \ln v_K^t - \Delta \ln P_A^t = \bar{\phi}_S (\Delta \ln Q_S^t + \Delta \ln u_S^t) + \bar{\phi}_L (\Delta \ln Q_L^t + \Delta \ln u_L^t) - \Delta \ln P_A^t + O(\Delta^2). \quad (\text{A.3})$$

The composite price index P_A^t is defined implicitly by $P_A^t Q_A^t = v_K^t$ and $Q_A^t = Q_A^{t-1} \cdot$ (quantity chain). Under the Törnqvist approximation:

$$\Delta \ln P_A^t = \bar{\phi}_S \Delta \ln u_S^t + \bar{\phi}_L \Delta \ln u_L^t + O(\Delta^2). \quad (\text{A.4})$$

Substituting (A.3) and (A.4) into (A.1) and using $\bar{s}_k = \bar{s}_K \bar{\phi}_k$ for $k \in \{S, L\}$:

$$\begin{aligned}
B^t &= \bar{s}_K \Delta \ln Q_A^t - \bar{s}_S \Delta \ln Q_S^t - \bar{s}_L \Delta \ln Q_L^t \\
&= \bar{s}_K [\bar{\phi}_S \Delta \ln Q_S^t + \bar{\phi}_L \Delta \ln Q_L^t + \underbrace{\bar{\phi}_S \Delta \ln u_S^t + \bar{\phi}_L \Delta \ln u_L^t - \Delta \ln P_A^t}_{=0 \text{ by Törnqvist}}] - \bar{s}_S \Delta \ln Q_S^t - \bar{s}_L \Delta \ln Q_L^t + O(\Delta^2) \\
&= (\bar{s}_K \bar{\phi}_S - \bar{s}_S) \Delta \ln Q_S^t + (\bar{s}_K \bar{\phi}_L - \bar{s}_L) \Delta \ln Q_L^t + O(\Delta^2).
\end{aligned}$$

Since $\bar{s}_K \bar{\phi}_k = \bar{s}_k$ for $k \in \{S, L\}$ by definition of ϕ_k , the $\Delta \ln Q_k$ terms cancel and $B^t = O(\Delta^2)$ if the price weights ϕ_k^t were time-invariant. The non-vanishing bias arises from the variation in ϕ_k^t over time: using $v_K^t = v_S^t + v_L^t$ and the second-order Taylor expansion $\ln(x + \epsilon) = \ln x + \epsilon/x - \epsilon^2/(2x^2) + O(\epsilon^3)$,

$$\begin{aligned}
\Delta \ln v_K &= \ln(\bar{\phi}_S e^{\Delta \ln v_S} + \bar{\phi}_L e^{\Delta \ln v_L}) + O(\Delta^2) \\
&= \bar{\phi}_S \Delta \ln v_S + \bar{\phi}_L \Delta \ln v_L - \frac{1}{2} \bar{\phi}_S \bar{\phi}_L (\Delta \ln v_S - \Delta \ln v_L)^2 + O(\Delta^3).
\end{aligned}$$

Therefore

$$\bar{\phi}_S \Delta \ln v_S + \bar{\phi}_L \Delta \ln v_L - \Delta \ln v_K = \frac{1}{2} \bar{\phi}_S \bar{\phi}_L (\Delta \ln v_S - \Delta \ln v_L)^2 + O(\Delta^3),$$

giving the second-order expression (6.6) after multiplying by \bar{s}_K . ■

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